SPICE Simulation of Transmission Lines by the Telegrapher's Method

Part 2: Putting Frequency Dependence into the Simulation

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Curve fitting of the primary parameters

As pointed out in Part 1, if we knew R_ω, L_ω, G_ω, and C_ω we would be ready to start our simulation. Let's look at how the primary parameters vary as a function of frequency. See the first four columns of **Table 1** below for values of the primary parameters for a typical twisted pair telephone cable. (A more extensive version of this table and tables for other gauges and temperatures can be found in Reference 3 and Reference 4. See References 5, 6, or 7 for an explanation as to why the parameters vary with frequency and how to go about deriving the equations for that frequency dependence.)

The general behavior of the primary parameters versus frequency is known.

Capacitance is dependent only on the dielectric constant of the insulation. This is a consequence of the fact that the conductor surface can be regarded as an equal potential surface up to very high frequencies and, as long as the wave length of the signal in the insulation is long with respect to the thickness of the insulation, the shape of the electric field is independent of frequency.

The ratio $\omega C_\omega / G_\omega$ is related to loss tangent, which is known to vary slowly with frequency.

Resistance is a constant at dc and low frequency. At high frequency it increases proportionately to the square root of the frequency as a consequence of skin effect and proximity effect.

Inductance is constant at dc and low frequency and is constant, but smaller at very high frequency. It is known to have the form of a sum of two parts. One part is a constant. The other part decreases inversely of the square root of frequency at high frequency due to skin effect.

	Primary Parameter Data				Primary Parameter Approximations				Primary Parameter Errors		
Freq	R	L	G	C	R	L	G	C	R	L	G
Hz	Ω/Kft	mH/Kft	μS/Kft	nF/Kft	Ω/Kft	mH/Kft	μS/Kft	nF/Kft	Ω/Kft	μH/Kft	nS/Kft
1	52.50	0.1868	0.000	15.72	52.50	0.1868	0.000	15.72	0.00	0.00	-0.05
10	52.50	0.1868	0.000	15.72	52.50	0.1868	0.000	15.72	0.00	0.00	-0.40
100	52.50	0.1868	0.003	15.72	52.50	0.1868	0.003	15.72	0.00	0.01	0.06
500	52.50	0.1867	0.012	15.72	52.50	0.1868	0.012	15.72	0.00	-0.06	0.08
1000	52.51	0.1867	0.022	15.72	52.50	0.1867	0.022	15.72	0.01	-0.02	0.22
2000	52.52	0.1866	0.040	15.72	52.50	0.1866	0.040	15.72	0.02	-0.03	0.19
5000	52.55	0.1863	0.088	15.72	52.51	0.1864	0.088	15.72	0.04	-0.08	-0.34
10 K	52.64	0.1859	0.162	15.72	52.56	0.1859	0.161	15.72	0.08	-0.05	0.54
20 K	52.91	0.1850	0.295	15.72	52.74	0.1851	0.295	15.72	0.17	-0.07	-0.09
50 K	54.32	0.1814	0.655	15.72	53.93	0.1824	0.655	15.72	0.39	-1.00	0.12
100 K	58.41	0.1770	1.197	15.72	57.64	0.1782	1.197	15.72	0.77	-1.20	0.10
200 K	69.89	0.1721	2.188	15.72	67.98	0.1716	2.188	15.72	1.91	0.53	0.46
300 K	81.73	0.1683	3.113	15.72	78.81	0.1669	3.113	15.72	2.92	1.39	0.16
500 K	102.59	0.1623	4.855	15.72	98.37	0.1609	4.855	15.72	4.22	1.36	0.26
1 M	141.30	0.1543	8.873	15.72	136.95	0.1537	8.873	15.72	4.35	0.62	0.15
2 M	196.03	0.1482	16.217	15.72	192.88	0.1479	16.217	15.72	3.15	0.25	0.38
5 M	304.62	0.1425	35.989	15.72	304.62	0.1426	35.989	15.72	0.00	-0.08	0.00

Table 1. Primary Parameters for 24-gauge telephone cable.

The following constants are from Table 1:

$\boldsymbol{\omega_2}$ = highest frequency in Table 1 (radians per second)
$\boldsymbol{\omega_1}$ = second highest frequency in Table 1
C_{dc} = capacitance at lowest frequency (nanofarads per 1000 feet)
G_{dc} = conductance at lowest frequency (micro Siemens per 1000 feet)
G_2 = conductance at $\boldsymbol{\omega_2}$
G_1 = conductance at $\boldsymbol{\omega_1}$

R_{dc} = resistance at lowest frequency (ohms per 1000 feet)
R_{ac} = resistance at ω_2
L_{dc} = inductance at lowest frequency (millineries per 1000 feet)

The values of the constants are:

ω_2 = $2\ \pi \times 5 \times 10^6$ rad/sec
ω_1 = $2\ \pi \times 3 \times 10^6$ rad/sec
C_{dc} = 15.72 nF/Kft
G_{dc} = 0.0005 $\mu\Omega^{-1}$/ Kft (or a small positive conductance for dc purposes).
G_2 = 35.989 $\mu\Omega^{-1}$/ Kft
G_1 = 16.217 $\mu\Omega^{-1}$/ Kft
L_{dc} = 0.1868 mH/Kft
R_{dc} = 52.50 Ω/Kft
R_{ac} = 304.62 Ω/Kft

In the table, G at the lowest frequency is 0.000 $\mu\Omega^{-1}$/ Kft. We need a positive number because SPICE may starts each run with a dc operating point analysis and letting Gdc be zero will result in an attempt to divide by zero. So, we choose for Gdc the largest value that would round down to 0.000 $\mu\Omega^{-1}$/ Kft which is 0.0005 $\mu\Omega^{-1}$/ Kft.

Equations of the following form match the known behavior at low and high frequencies and fit the data from Table 1 well.

3a. $C_\omega = C_{dc}$

This one is obvious. The data in the table is independent of frequency and so is this equation.

3b. $$R_\omega = R_{dc} \left(1 + \left(\frac{\omega}{\omega_R} \right)^2 \right)^{1/4}$$

3c.

$$\omega_R = \omega_2 \times \frac{R_{dc}^{\ 2}}{(R_{ac}^{\ 4} - R_{dc}^{\ 4})^{1/2}} \approx 933{,}562$$

For $\omega \ll \omega_R$, this is obviously a constant equal to R_{dc}. For high-enough frequency, it increases proportionately to the square root of frequency. We select R_{dc} as the resistance at the lowest frequency available and then we select ω_R to make this equation agree with Table 1 at the highest available frequency.

4a.

$$G_\omega = G_{dc} + G_2\left(\left(\frac{\omega}{\omega_2} \right)^2 \right)^K$$

4b.

$$K = \left(\frac{1}{2}\right) \frac{\log\left(\frac{G_2}{G_1}\right)}{\log\left(\frac{\omega_2}{\omega_1}\right)} = 0.435$$

Obviously, if k = 0.5, then the ratio $\omega C_\omega / G_\omega$ varies slowly (none at all in fact). We will select G_2 to be the conductance at the highest frequency in the table and ω_2 to be the highest frequency in the table. By the way the equation is set up, it must be exactly correct at the highest frequency in the table. We will select k so that the equation is also exactly correct for the second-highest frequency in the table. Upon evaluation, the equation is found to be in error by less than 0.01 $\mu\Omega^{-1}$/ Kft at all frequencies.

$$L_\omega = L_\infty + \frac{(L_{dc} - L_\infty)}{\left(1 + A\left(\frac{\omega}{\omega_L}\right) + \left(\frac{\omega}{\omega_L}\right)^2\right)^{1/4}}$$

5a.

where:

L_∞	=	133.0 μH/Kft
A	=	1.6
ω_L	=	2 x π x 161000

If $\boldsymbol{\omega} \ll \boldsymbol{\omega_L}$, then the equation is equal to L_{dc}. If $\boldsymbol{\omega} \gg \boldsymbol{\omega_L}$ then the variable part of the inductance decreases inversely to the square root of frequency. The worst error is 2.2 μH at 500 kHz.

There are several combinations of L_∞, A, ω_L that give equal results. These values were found by trial and error. Given several combinations with equal accuracy, then favor the one that minimizes the difference between L_{dc} and L_∞.

Laplace transforms are expressed as functions of s. During frequency domain simulation, s is replaced by $j\boldsymbol{\omega}$ and evaluated. "j" is the square root of -1. However, the functions for R_ω, L_ω, G_ω, and C_ω vs. $\boldsymbol{\omega}$ are real valued functions of a real variable. Noting that $-s^2 = \boldsymbol{\omega}^2$ is real and positive, every occurrence of $\boldsymbol{\omega}$ in R_ω, L_ω, G_ω, and C_ω will be replaced with $(-s^2)^{1/2}$. In cases of $\boldsymbol{\omega}^n$ where n is even, ω^n can be replaced with $(-s^2)^{n/2}$.

The accuracy of our curve fitting attempts are shown in Table 1. The first four columns are the given data. The next 4 columns are the results of evaluation of our approximating equations. The last three columns give the errors. Errors for capacitance have been omitted since they are identically zero. Note that the inductance and conductance errors are in smaller units than the corresponding data entries.

Of course, you don't really care how well you match the ohms per kilo foot. What you care about is the dBs per kilofoot and nanoseconds per kilofoot. In other words, we want to know how well we match the secondary parameters. The secondary parameters are shown in **Table 2**. The secondary parameters can be computed from the primary parameters and vice versa. The last secondary parameter, the delay per unit length, can also be expressed as phase shift per unit length or velocity.

	Secondary Parameters from Data				Secondary Parameter Approximations				Secondary Parameter Errors			
Freq	Z		α	delay	Z		α	delay	Z		α	delay
Hz	Ω	°	dB/Kft	µs/Kft	Ω	°	dB/Kft	µs/Kft	mΩ	°	dB/Kft	ns/Kft
1	23055	-45.0	0.01	256.3	23055	-45.0	0.01	256.2	2	-0.02	0.00	69.3
10	7291	-45.0	0.04	81.05	7291	-45.0	0.04	81.03	0	-0.01	0.00	16.3
100	2305	-44.9	0.14	25.65	2305	-44.9	0.14	25.65	0	0.00	0.00	-0.1
500	1031	-44.7	0.31	11.52	1031	-44.7	0.31	11.52	-1	0.00	0.00	0.0
1000	729.22	-44.4	0.44	8.19	729.16	-44.4	0.44	8.19	65	0.00	0.00	0.7
2000	515.88	-43.7	0.61	5.86	515.79	-43.7	0.61	5.86	86	0.00	0.00	0.9
5000	327.21	-41.8	0.94	3.83	327.10	-41.8	0.94	3.83	107	0.00	0.00	1.1
10 K	233.65	-38.7	1.25	2.86	233.48	-38.7	1.25	2.86	169	-0.01	0.00	1.6
20 K	171.04	-33.1	1.60	2.25	170.81	-33.1	1.60	2.25	230	-0.04	0.00	2.0
50 K	126.26	-21.8	2.01	1.84	126.23	-21.6	2.00	1.84	34	-0.18	0.02	-1.8
100 K	112.77	-13.9	2.32	1.72	112.92	-13.6	2.28	1.73	-141	-0.23	0.04	-3.9
200 K	107.26	-8.95	2.87	1.67	106.98	-8.75	2.79	1.66	287	-0.20	0.07	3.5
300 K	105.15	-7.22	3.40	1.64	104.62	-7.03	3.30	1.63	524	-0.19	0.11	7.5
500 K	102.62	-5.69	4.37	1.61	102.13	-5.50	4.20	1.60	496	-0.18	0.16	7.3
1 M	99.60	-4.14	6.18	1.56	99.37	-4.03	6.00	1.56	227	-0.11	0.18	3.3
2 M	97.36	-3.00	8.76	1.53	97.27	-2.96	8.63	1.53	90	-0.04	0.13	1.4
5 M	95.32	-1.94	13.90	1.50	95.34	-1.94	13.90	1.50	-25	0.00	0.00	-0.4

Table 2. Secondary Parameters

Expressing the primary line parameters in Laplace Transform notation yields:

6a. $$C_s = C_{dc}$$

6b. $$R_s = R_{dc} \left(1 - \left(\frac{s}{\omega_R} \right)^2 \right)^{1/4}$$

6c. $$G_s = G_{dc} + G_2\left(-\left(\frac{s}{\omega_2} \right)^2 \right)^K$$

6d. $$L_s = L_\infty + \frac{(L_{dc} - L_\infty)}{\left(1 + A\left(-\left(\frac{s}{\omega_L} \right)^2 \right)^{1/2} - \left(\frac{s}{\omega_L} \right)^2 \right)^{1/4}}$$

We will insert these equations into the expressions for Z(s) and F(s).

Other approximations

The approximations that we have used are merely convenient and are not unique. There are instances in the literature, such as **Reference 8**, where the primary parameters are expressed as other functions of frequency. These functions can be used instead of the approximating functions introduced in this article.

There are a few points to check though. The function must be a function of ω which is in radians per second. Reference 8 offers up its approximations as functions of KHz.

There may be a necessity to convert the units. For example, on inductance, reference 3 uses mH/Kft, reference 4 uses mH/mile and reference 8 uses μH/km.

Extensibility to higher frequency

Is reasonable to presume that these simple functions for the primary parameters can be extended beyond the highest frequency in the table? The answer is yes, probably for C, R, and G because curves fit very well and in the case of R, the data it is well into the skin effect limited region of the curve. However in the case of L the data does not include high enough frequencies to reach the part of the curve where it flattens out at high frequency. We don't know L_∞, the high frequency limit for inductance. And since the high frequency inductance sets the high frequency characteristic impedance per the equation:

$$Z_\infty = \sqrt{\frac{L_\infty}{C_\infty}}$$

7a.

For the best results, it is desirable to know L_∞. If you have other knowledge of Z_∞ and use that to compute L_∞ then the model might work well at higher frequencies. Otherwise, no promises!

In the last part of this article we will put those approximating equations into the two-model for a transmission line and produce a sub-circuit that is ready to use.

References

1. *Engineering Electromagnetics*, 4th edition by William Hayt, 1981. Chapter 12, section 1, "The Transmission Line Equations".
2. Wikipedia.org, "Primary line constants", "telegrapher's equations" and "propagation constant".
3. *Subscriber Loop Signaling and Transmission Handbook*, by Whitman D. Reeve, 1995, IEEE Press.
4. *DSL Simulation Techniques and Standards Development for Digital Subscriber Line Systems*, by Walter Y. Chen, 1998, Macmillan Technical Publishing.
5. *Engineering Electromagnetics*, 4th edition by William Hayt, 1981. Chapter 12, section 3, "Transmission-Line Parameters".
6. *Electric Transmission Lines*, by Skilling, 1951. Chapter 7.
7. *Fields and Waves in Communication Electronics*, by Ramo, Whinnery and van Duzer, 1965. Chapter 5.

8. *Home Networking Basis: Transmission Environments and Wired/Wireless Protocols*, Chapter 2, section 1.2, by Walter Y. Chen, 2003, Prentice Hall.

Resources

1. Linear Technology Corporation LT Spice IV download available with a very generous license.
   http://www.linear.com/designtools/software/ltspice.jsp
2. Yahoo user's group
   http://tech.groups.yahoo.com/group/LTspice/

About the author

Roy McCammon is a senior engineer with 3M's Communication Markets Division, and a graduate of the University of Texas Department of Electrical Engineering. He has operated a satellite tracking station in Antarctica, designed astronomical instruments and telescope servos, and has spent the last 28 years designing test equipment used by telecommunications providers and thinking about transmission lines. His interests include dancing the two-step, pushing miniature battleships around a table top, and simulations.