Morphing the Circuit

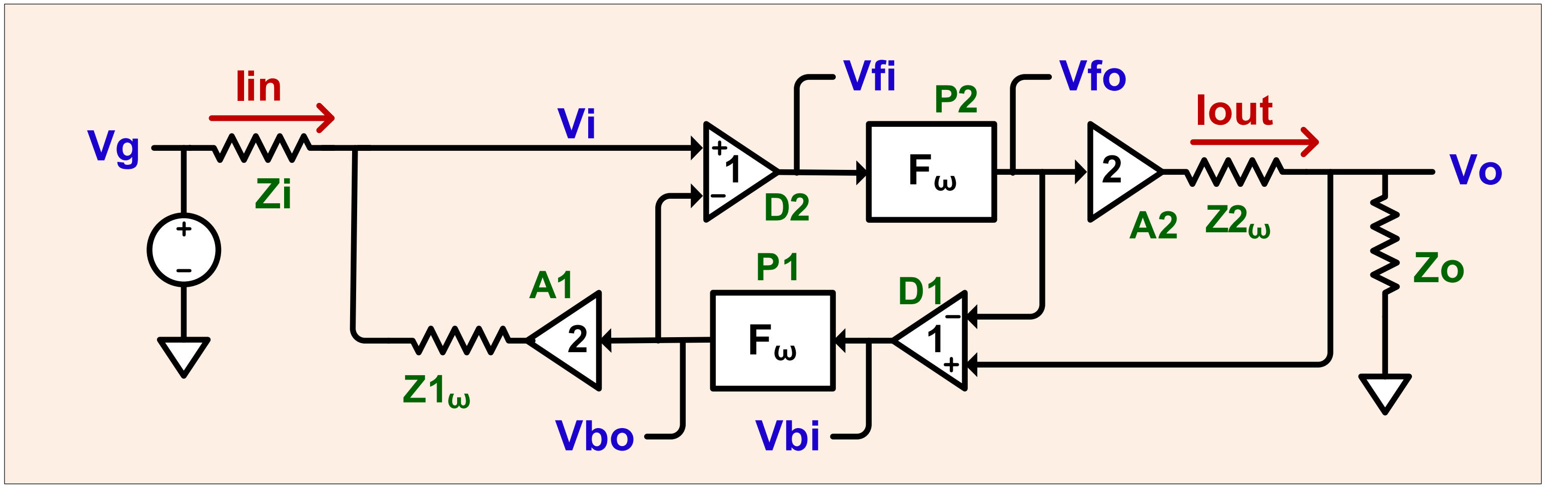


Figure 1:

While we could use the circuit shown in figure 1 directly, it is not optimal for SPICE.
SPICE has some idiosyncrasies:

Current sources are better than voltage sources.

Laplace transform elements work better if they approach zero at high frequencies.

An inductor with an internal series resistance works better than a separate inductor and resister.

SPICE works better at simulating balanced circuits if the simulated circuit is balanced.

The first three are documented, the last is not, but has been observed. We speculate that certain elements in the sparse matrix cancel exactly in one case and not in the other.

Currents are proportional to voltages. Voltage sources can be replaced with current sources. Voltage dependent sources can be replaced with current dependent sources. By applying some circuit theory transformations, the circuit in figure 1 can be converted to the circuit in fig. 12.

Step 1: redraw the circuit into the more symmetric form shown in fig. 2.

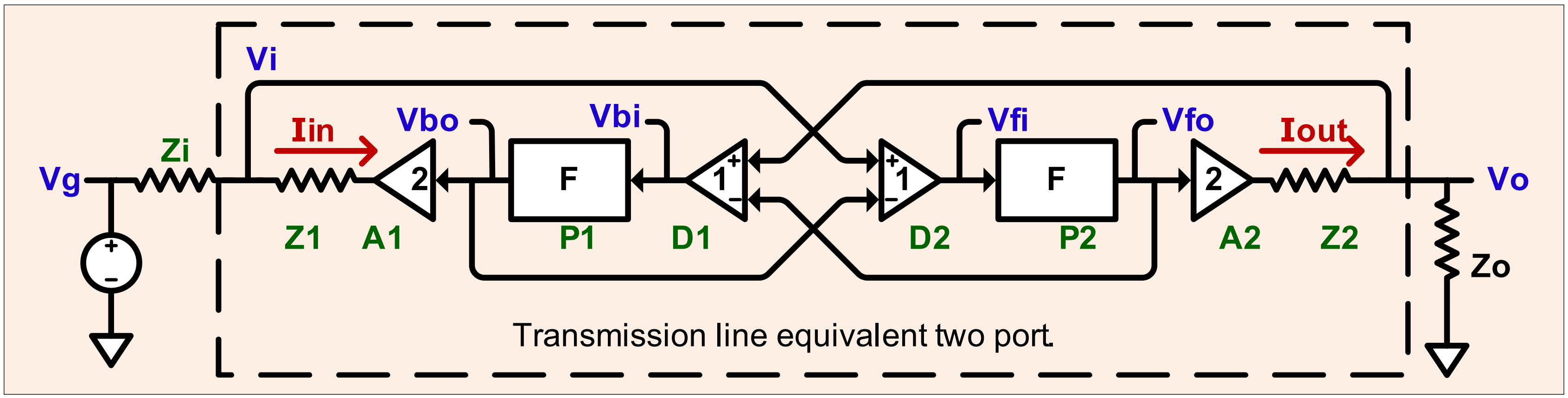


Figure 2: The circuit redrawn to emphasize the symmetry.

Step 2: replace voltage doubler A1 and series impedance Z1 with double current sources G3 and G5 and Z1 in shunt. Likewise A2 and Z2 in series have been replacing by G4 and G6 and Z2 in shunt. The trans-conductance of G3 through G6 is 1/Z. The result is shown in fig 3. The reason for using double current sources instead of one current source with twice the gain will become apparent in a few steps.

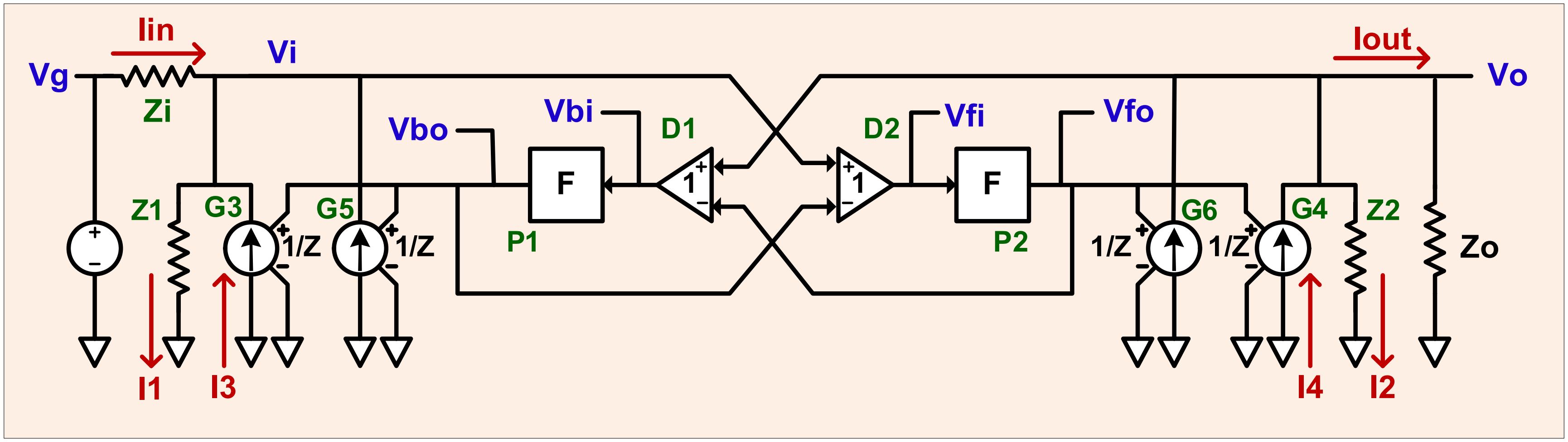


Figure 3. Voltage doublers replaced by transconductance amplifiers.

Step 3: replace the difference amplifier D2 by transimpedance amplifiers H2 and H4 in series. D2 produced the output

$$Vfi = Vi - Vbo.$$

It should be obvious that

$$I1 = Vi / Z$$
$$I3 = Vbo / Z.$$

Solving for the voltages yields

$$Vi = (Z)\ I1$$
$$Vbo = (Z)\ I3$$

The series combination of H2 and H4 simply produces Vfi = Vi – Vbo, which is the same as the output of D2. Likewise, D1 has been replaced by the series combination of H1 and H3. The result is shown in fig 4.

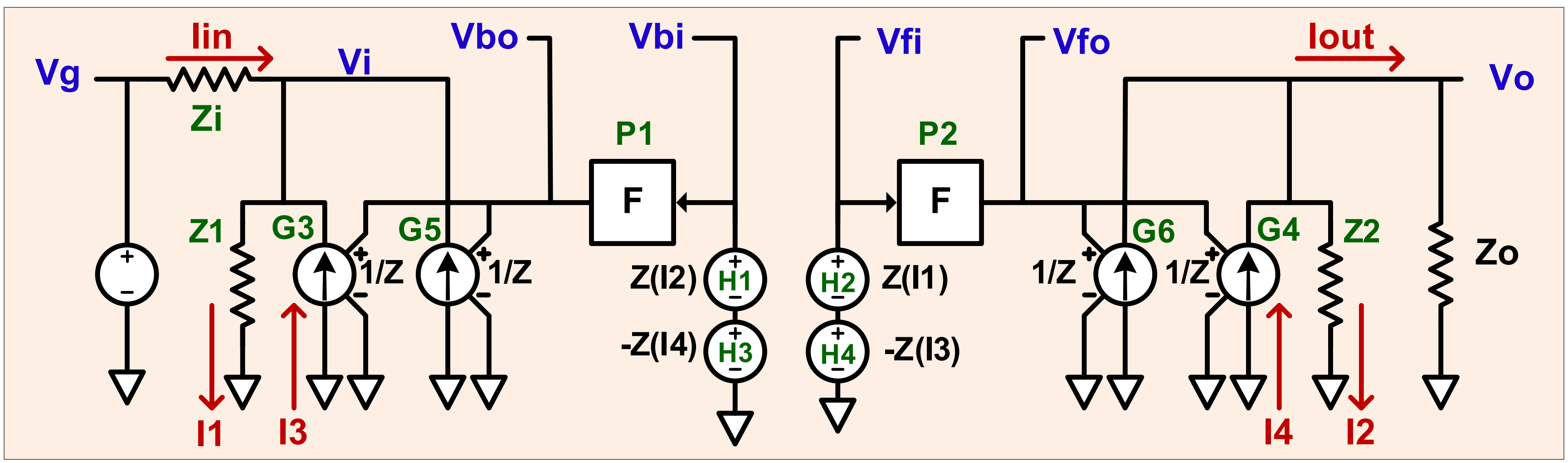


Figure 4. Difference amplifiers replaced by transimpedance amplifiers.

Step 4: insert a zero volt source, V6, into a circuit branch that happens to carry the current I2-I4. The series combination of H1 and H3 produces the signal I2-I4. A single transimpedance amplifier, H6, controlled by I6, the current through V6, can replace the series combination of H1 and H3. Similarly, H5 replaces H2 and H4. The reason for the doubled current sources instead of a current source with twice the gain is so that the branch currents I1 - I3 and I2 - I4 exist. The voltages Vbo, Vbi, Vfi and Vfo are no longer used in any expression and will be suppressed.

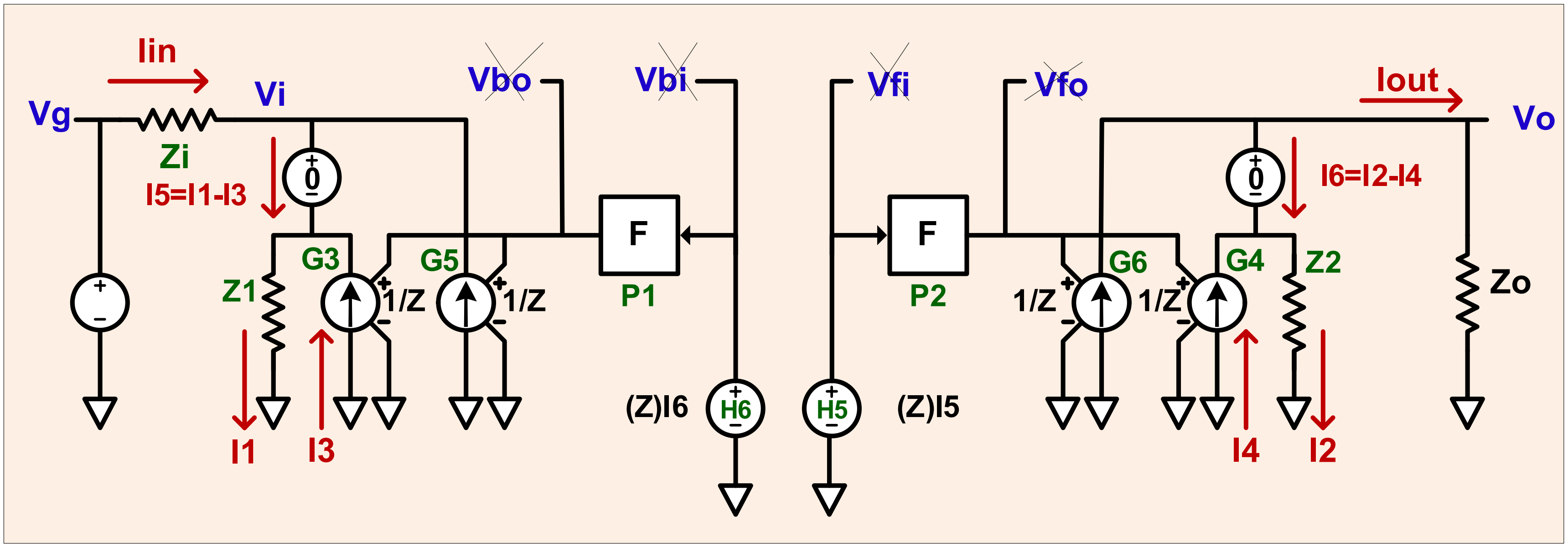


Figure 5. Voltage difference replaced with current difference.

Step 5: decrease the transimpedance gain of H1, H2, H3 and H4 by Z to 1 ohm and increase the transconductance gain of G3, G4, G5 and G6 by Z to 1/ohm. The gain from H6 to Vi is unchanged as is the gain from H5 to Vo. The gain of G3, G4, G5 and G6, which is now unity, will be suppressed in the following figures. The result is shown in fig 6.

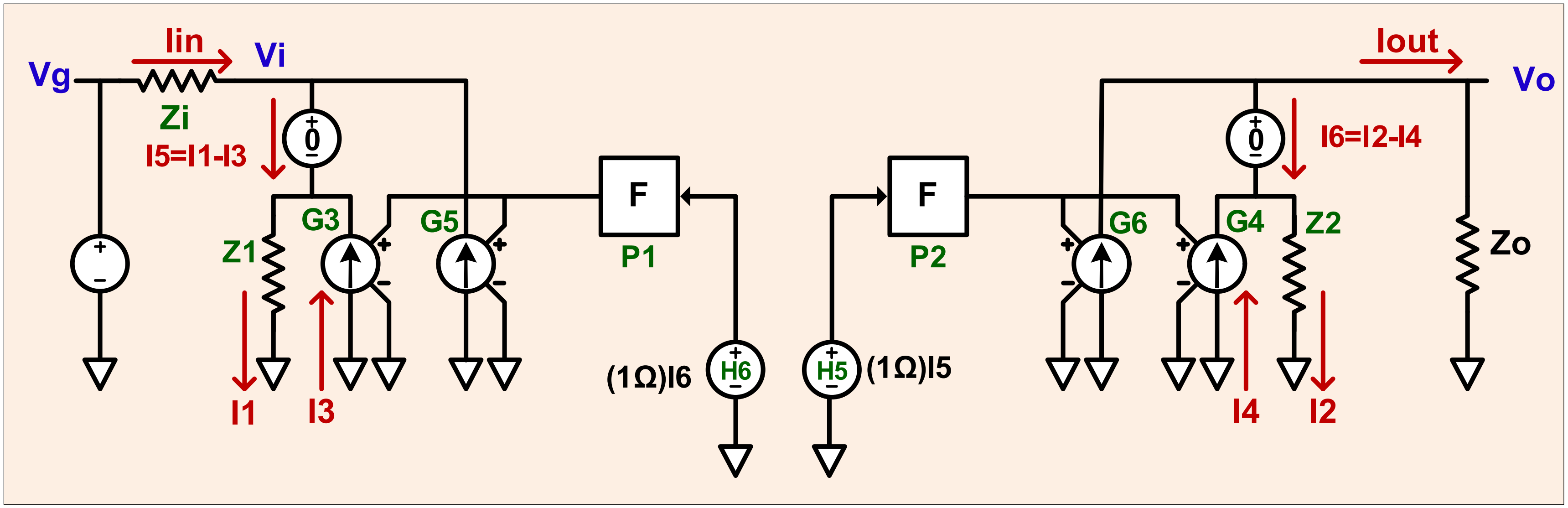


Figure 6. Gains normalized.

Step 6: replace the propagation function P1 by transconductance amplifier G7 driving a 1 ohm resister. Replace P2 by G8 and a 1 ohm resister. The result is shown in figure 7.

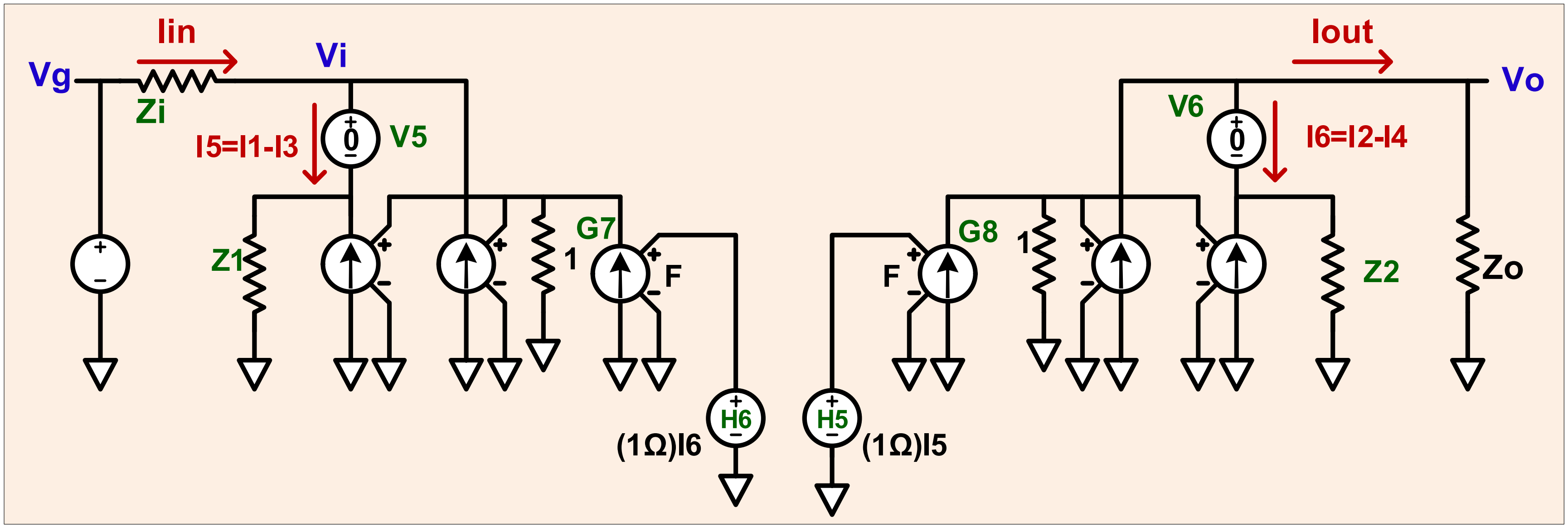


Figure 7. Propagation functions replaced with transconductance amplifiers.

Step 7: replace the impedance Z1 by transconductance amplifier G1. Likewise replace Z2 G2.

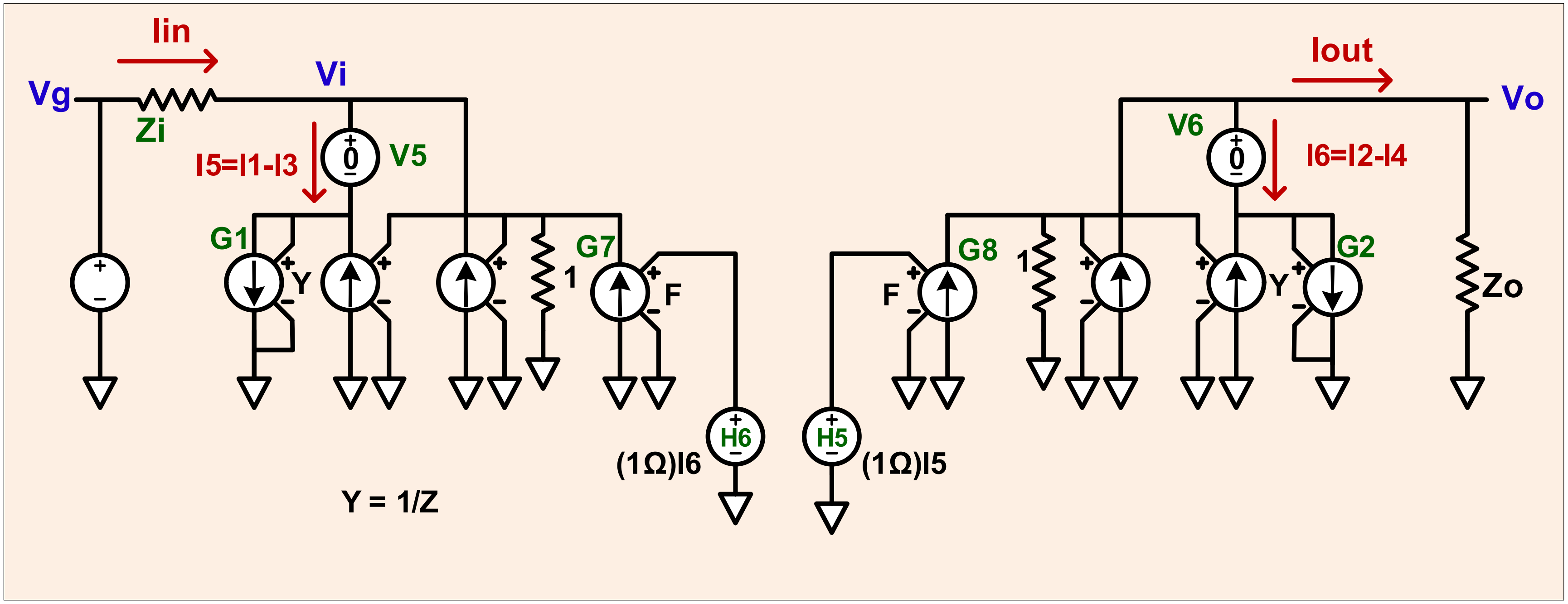


Figure 8. Characteristic impedences replaced with transconductance amplifiers.

There are two problems with the circuit of fig. 8. SPICE requires that Laplace elements roll off to zero at high frequency. G1 and G2 approach a constant. G7 and G8 do go to zero, but not quickly enough.

Step 8 Increase the roll-off of G7 by dividing its Laplace transform by (1 + s Lcon). Restore the frequency response by adding an inductor, Lcon, in series with the 1 ohm resister that G7 drives. Do the same thing to G8 and the 1 ohm resister that it drives.

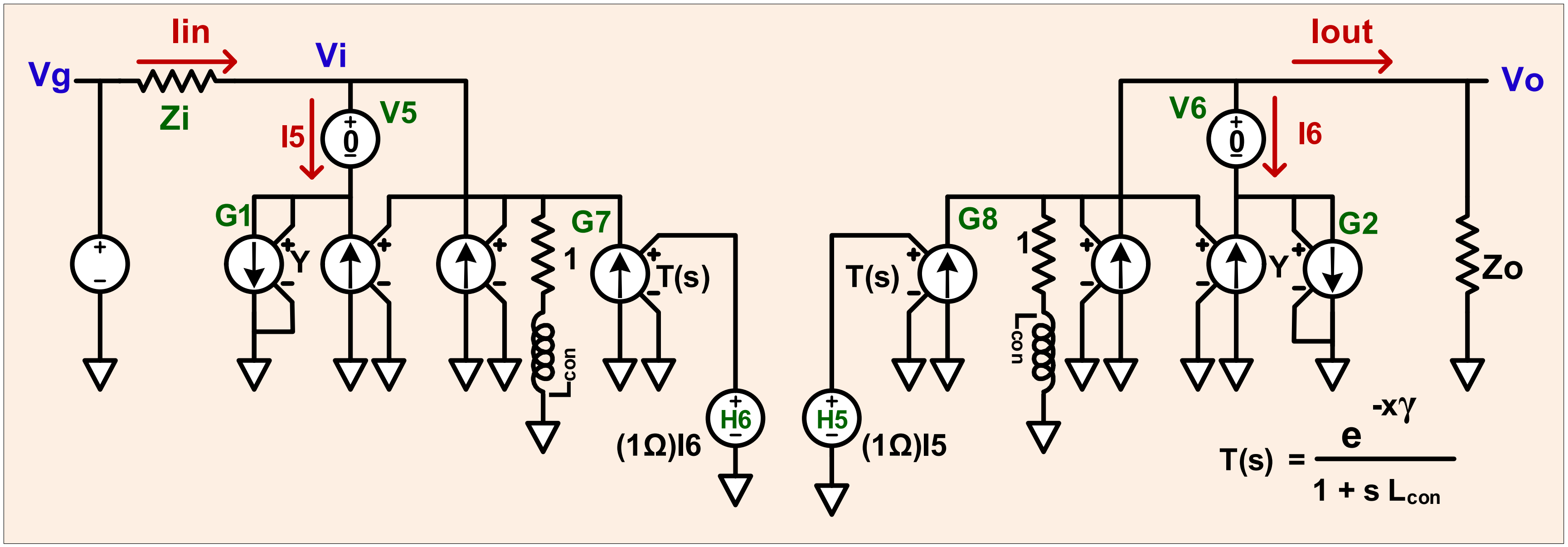


Figure 9.

Step 9: Since, G1 in the previous figure approaches a constant, subtract that constant out of G1. What is left will go to zero at high frequency. The constant removed happens to be a real valued conductance. Add that conductance back by adding a simple resister in shunt with G1. Likewise, subtract the same constant out of G2 and add it back as a shunt resister.

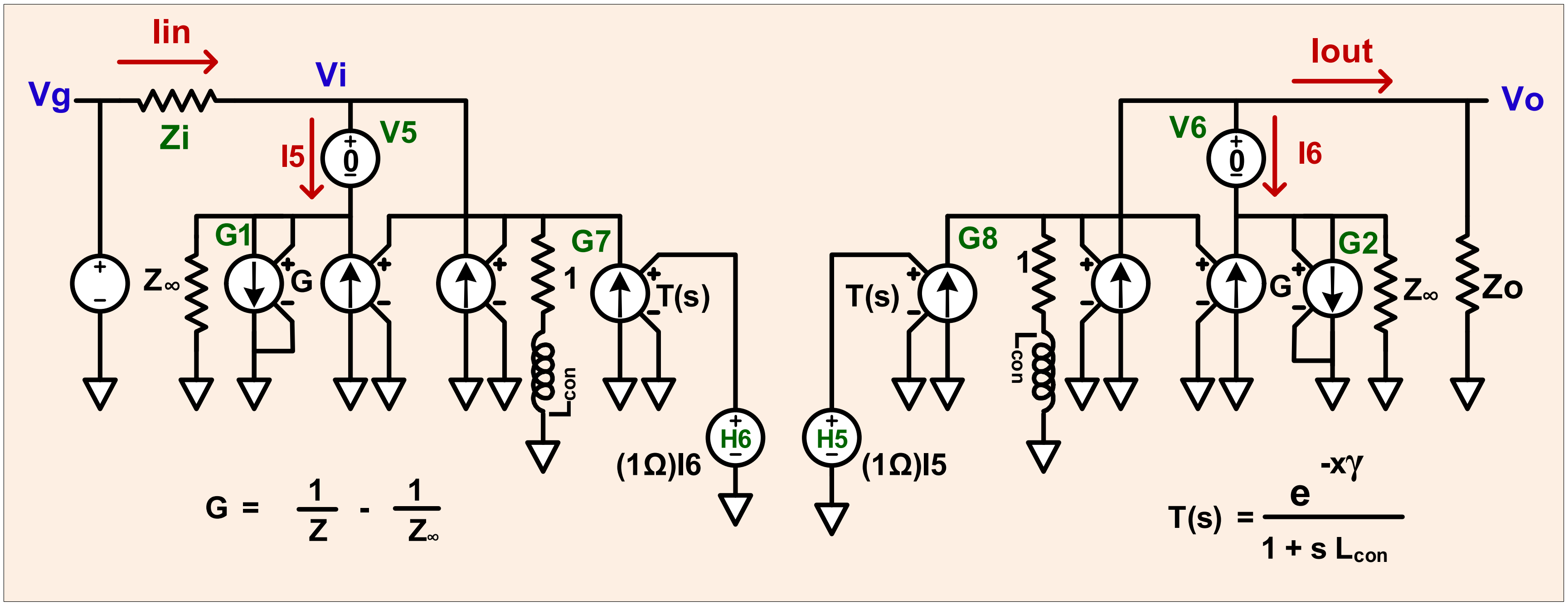


Figure 10.

Step 10: absorb the 1Ω resister that was in series with Lcon into Lcon as an internal series resistance.

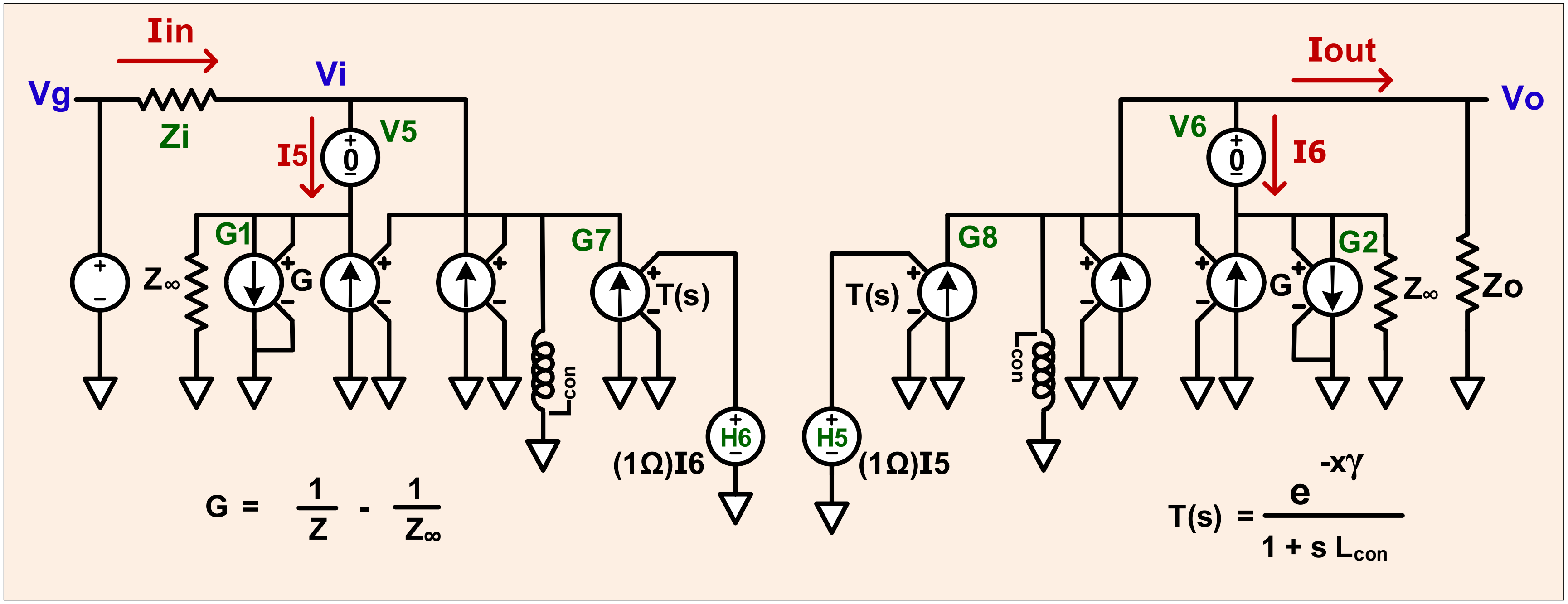


Figure 11

This circuit is now SPICE friendly and can be simulated directly, but it is not balanced. If you are simulating an unbalanced trans,ission line, like a coaxial cable, then you are done. If you simulating a balanced transmission line like a twisted pair, there are a few more steps.

Step 11: change the input source from an unbalanced voltage source to a balanced current source driving a split resister with the center point grounded. The load resistance has also been split with the center point grounded. The load and source have been grounded because SPICE requires a path to ground for all nodes.

G8 gets its entire signal from V5 which does not occupy a balanced place in the circuit. G7 gets its entire signal from V6 which likewise does not occupy a balanced position in the circuit. Add V7 as a mirror image to V5 and get half of the signal for G8 from V5 and half from V7. Likewise add V8 as a mirror to V6 and get half the signal for G7 from V6 and half from V8. (Dear reader, we haven't dreamed this up as an academic exercise; we had a real problem that would not simulate properly until we made this modification).

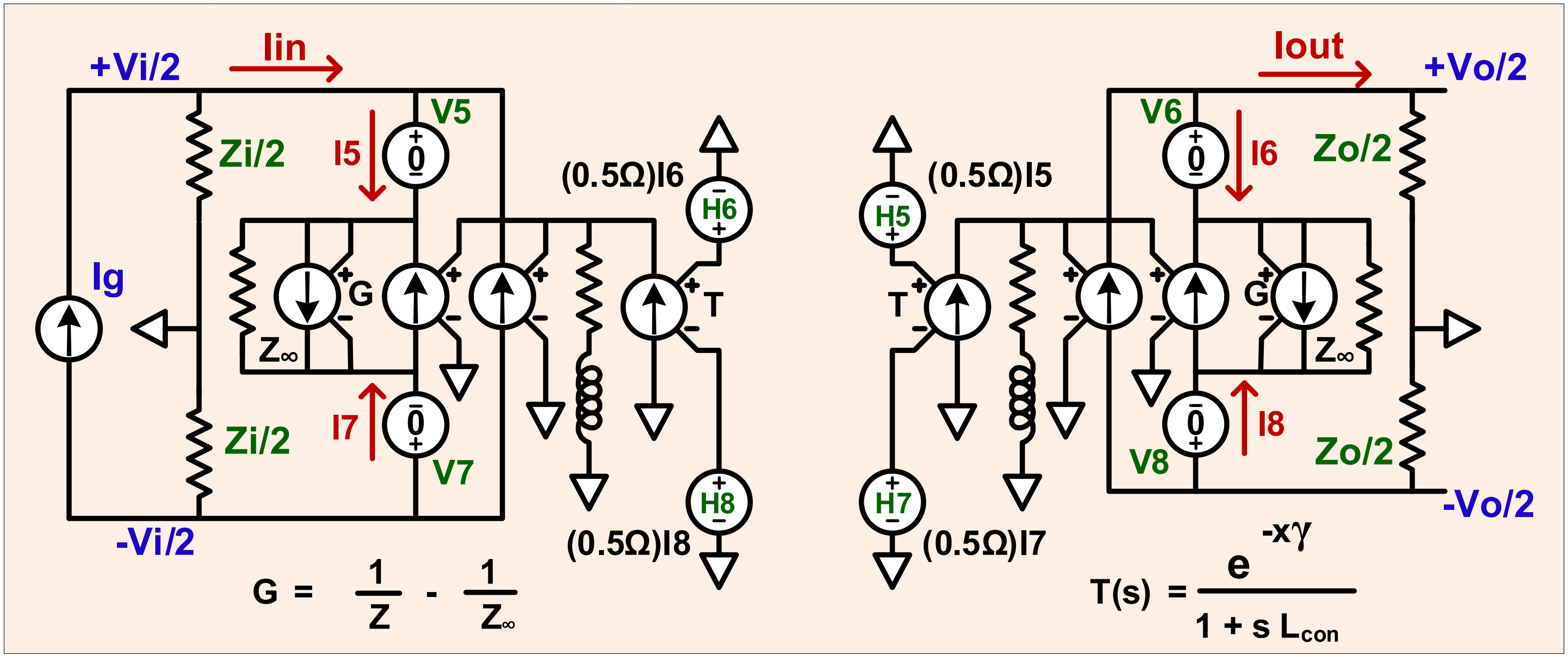


Figure 12

The circuit is now in its final, balanced SPICE ready form. But SPICE doesn't use the mathematical symbols that we have been using. Here then, are the parameter statements and Laplace expressions in their actual SPICE form.

This is the parameter statement that establishes the constants. Note, in a .param statement, ** means exponentiation.

```
.param
+ Kft=1  ; 1000 feet
+ Lcon=1n
+ C=15.72e-9   Gdc=0  Rdc=52.50   Ldc=0.1868e-3      ; dc values
+ F2=5e6       G2=36u Rac=304.62  W2=6.28318*F2      ; highest frequency values
+ F1=3e6       G1=23u                                ; second highest frequency values
+ WR=W2*(Rdc**2)/(((Rac**4)-(Rdc**4))**0.5)          ; resistance term
+ k=Log(G2/G1)/Log(F2/F1)/2                          ; conductance exponent
+ Zinf=(Linf/C)**0.5        Yinf=1/Zinf
+ Ldel=(Ldc-Linf)
+ Linf= 0.133e-3    A=1.6     wL=6.28318*161000      ; inductance terms
```

This is the Laplace expression for G7 and G8. Note, in a Laplace expression, ^ means exponentiation.

```
Laplace=Exp(-Kft*((((Rdc*(1-(s/wR)^2)^.25)+s*(Linf+Ldel/(1+A*((-(s/wL)^2)^.5)-(s/wL)^2)^.25))*(Gdc+G2*( -
(s/w2)^2)^k+s*C))^.5))/(s*Lcon+1)
```

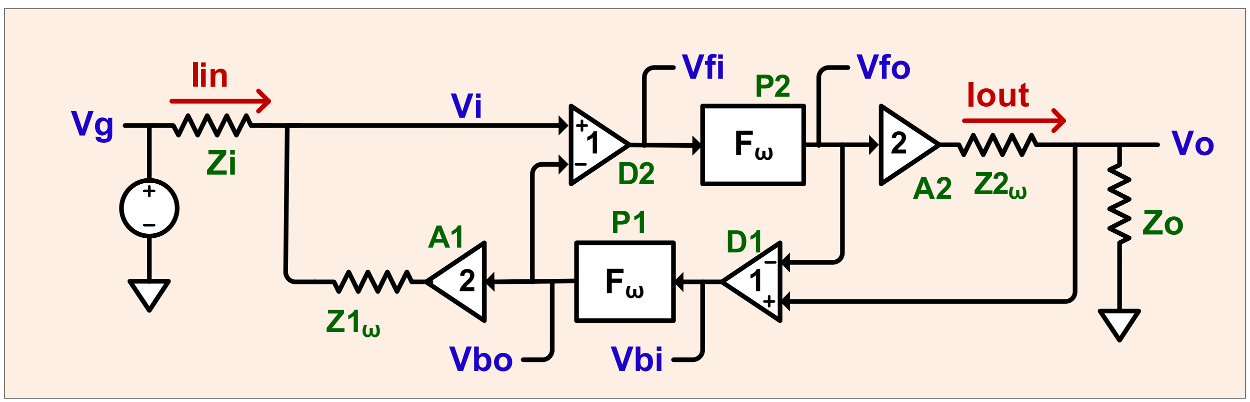
This is the Laplace expression for G1 and G2.

```
Laplace=(((((Gdc+G2*(-(s/w2)^2)^k)+s*C)/((Rdc*(1-(s/wR)^2)^0.25)+s*(Linf+Ldel/(1+A*((-(s/wL)^2)^0.5)-(s/wL)^2)^0.25)))^0.5)) -Yinf
```

The Transfer Function of a Transmission Line with Arbitrary Load and Source Impedances

Solving a circuit from the inside out

Usually, we assume an input voltage and then solve for the other node voltages. After that linearity assures us the all the voltages are proportional to the input voltage. But there is nothing special about input node. You can start with any node and assume a voltage on it and then solve for all the other node voltages. This is especially useful when there is a loop in the signal path. If you work from the input you usually wind up with a set of simultaneous linear equations to solve. But sometimes you can pick a node on the loop and the problem simplifies.



Canonical two port equivalent of a transmission line.

For the circuit in fig. 1 we will select Vfi, the input to the P2 block, as the independent variable. From that we can directly compute all the node voltages and branch currents. If that doesn't seem mathematically sound, think of it this way: you have a variable source and a voltmeter. Your voltmeter is connected to Vfi and the variable source is connected at Vg.

The source has no voltage readout and the voltmeter can only readout Vfi. So, you vary the input source until the voltmeter reads 1 volt. You then apply circuit theory and you discover that in this case you can then compute all the node voltages and branch currents.

Once we have an equation for Vg as a function of Vfi, we can solve that equation for Vfi as a function of Vg. We can substitute that expression for Vfi in all the other equations for node voltage or branch current and we will have the traditional solutions without ever setting up or solving any simultaneous equations.

Definitions:

$\Gamma_1 \equiv (Zi - Z1) / (Z1 + Zi)$

$\Gamma_2 \equiv (Zo - Z2) / (Z2 + Zo)$

$\delta \equiv (1 - \Gamma_1 \Gamma_2 F^2)$

$Y_1 \equiv 1/Z1$

$Y_2 \equiv 1/Z2$

useful combinations

$(1 + \Gamma_1) = (2)(Zi) / (Zi + Z1)$

$(1 + \Gamma_2) = (2)(Zo) / (Zo + Z2)$

$(1 - \Gamma_2) = (2)(Z2) / (Z2 + Zo)$

$(1 - \Gamma_1) = (2)(Z1) / (Z1 + Zi)$

$Zi\, Y_1 = (1 + \Gamma_1) / (1 - \Gamma_1)$

The output of P2 is just F times its input.

$$Vfo = (F)Vfi$$

The output, Vo, is determined by the gain of A2 and the voltage divider created by Z2 and Zo.

$$\begin{aligned} Vo &= (2\ Vfo)\,[\,Zo / (Zo + Z2)\,] \\ &= (1 + \Gamma_2)(F)\ Vfi \end{aligned}$$

The output current, Iout, can be determined by the voltage drop across Z2.

$$
\begin{aligned}
Iout &= [(2Vfo) - Vo] / Z2 \\
&= [2(F)Vfi - (1 + \Gamma_2)(F)Vfi]\ Y_2 \\
&= (F)\ [2 - (1 + \Gamma_2)]\ Y_2\ Vfi \\
&= (F)\ (1 - \Gamma_2)\ Y_2\ Vfi
\end{aligned}
$$

The output of D1 is determined from its inputs Vo and Vfo.

$$
\begin{aligned}
Vbi &= Vo - Vfo \\
&= (1 + \Gamma_2)(F)Vfi - (F)Vfi \\
&= (\Gamma_2 F)Vfi
\end{aligned}
$$

The output of P1 is just F times its input.

$$
\begin{aligned}
Vbo &= (F)\ Vbi \\
&= (\Gamma_2 F^2)Vfi
\end{aligned}
$$

The output of D2 is determined from Vi and Vbo.

$$
Vfi = Vi - Vbo
$$

This can be solved for Vi.

$$
\begin{aligned}
Vi &= Vfi + Vbo \\
&= Vfi + (\Gamma_2)(F^2)Vfi \\
&= (\ 1 + \Gamma_2 F^2\ )\ Vfi
\end{aligned}
$$

The input current, Iin, can be determined by the voltage drop across Z1.

$$\begin{aligned}
Iin &= (Vi - 2Vbo)/Z1 \\
&= [(1 + \Gamma_2 F^2)Vfi - 2(\Gamma_2 F^2)Vfi]\, Y_1 \\
&= Y_1\, Vfi(1 - \Gamma_2 F^2)
\end{aligned}$$

Vg can be determined from Vi and Iin.

$$\begin{aligned}
Vg &= Vi + (Zi)Iin \\
&= (1 + \Gamma_2 F^2)\, Vfi + (1 - \Gamma_2 F^2)\, Zi\, Y_1\, (Vfi) \\
&= [(1 + \Gamma_2 F^2) + (1 - \Gamma_2 F^2)(1 + \Gamma_1) / (1 - \Gamma_1)]\, (Vfi) \\
&= 2(Vfi)(1 - \Gamma_2 F^2) / (1 - \Gamma_1)
\end{aligned}$$

This can be solved for Vfi

$$\begin{aligned}
Vfi &= (Vg/2)(1 - \Gamma_1) / (1 - \Gamma_1 \Gamma_2 F^2) \\
&= (Vg/2)(1 - \Gamma_1) / \delta
\end{aligned}$$

Substituting for Vfi in selected equations gives:

$$\begin{aligned}
Vi &= (Vg/2)\,(1 + \Gamma_2 F^2)\,(1 - \Gamma_1)/\delta \\
Vo &= (Vg/2)\,(1 + \Gamma_2)(F)(1 - \Gamma_1)/\delta \\
Vo/Vi &= (F)(1 + \Gamma_2)/(1 + \Gamma_2 F^2) \\
Iout &= (Y_2\, Vg/2)\,(F)\,(1 - \Gamma_2)\,(1 - \Gamma_1) / \delta \\
Iin &= (Y_1\, Vg/2)\,(1 - \Gamma_1)(1 - \Gamma_2 F^2)/\delta
\end{aligned}$$

We have now solved the circuit without ever solving or even creating simultaneous equations.

Interpretation

Let's see what these equations tell us in the context of transmitting brief pulses. We will assume that the transmission line is matched at source so $\Gamma_1 \equiv 0$ and $\delta \equiv 1$ and the equations of interest simplify to

$$Vi = (Vg/2)(1 + F^2\Gamma_2)$$
$$Vo = (Vg/2)(F)(1+\Gamma_2)$$
$$Iout = (Y_2 Vg/2)(F)(1 - \Gamma_2)$$

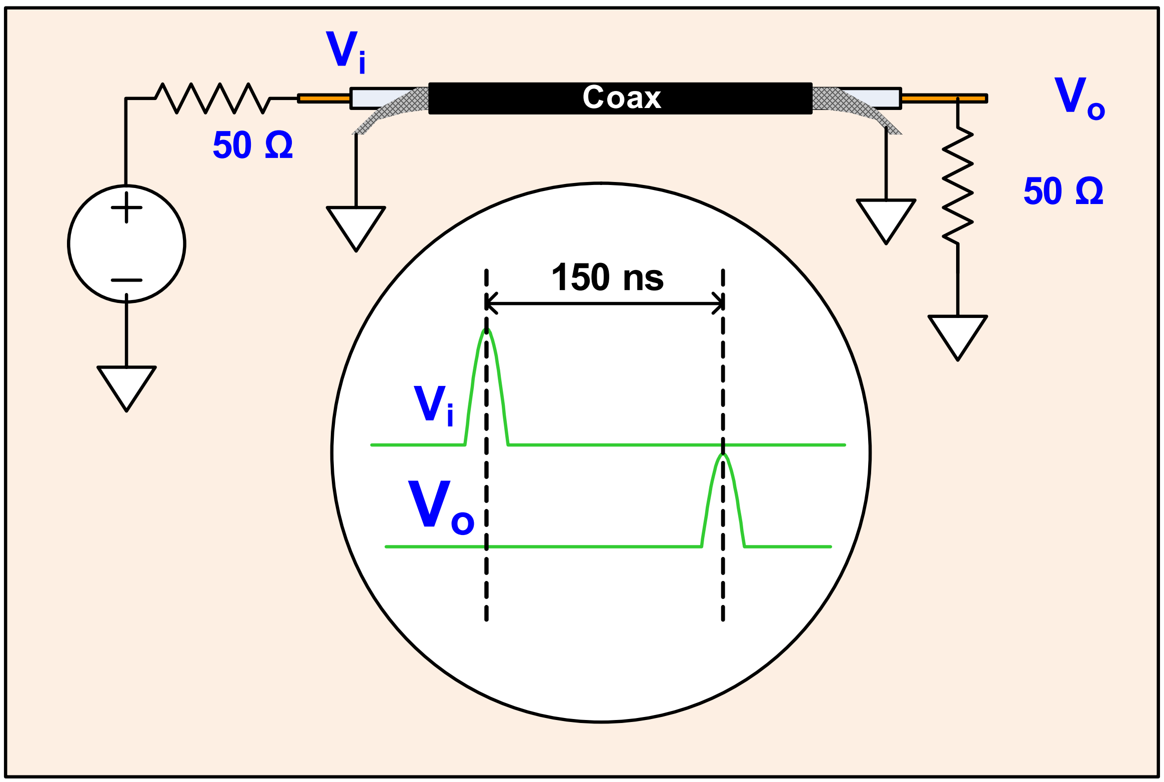
We will assume a 50 ohm transmission line that has a delay of 150 ns and reduces pulse amplitude to 80% in a one way transit. We will also assume Vg produces a 2 volt pulse. In this context:

$F \equiv$ a gain of 0.80 and a delay of 150 ns
$F^2 \equiv$ a gain of 0.64 and a delay of 300 ns
$Vg/2 \equiv$ a 1 volt pulse
$Y_2 Vg \equiv$ a 40 ma current pulse

Case 1. Coax is matched at the load, hence $\Gamma_2 = 0$

Vi	=	(Vg/2)
Vo	=	(Vg/2) (F)
Iout	=	(Y_2 Vg /2)(F)

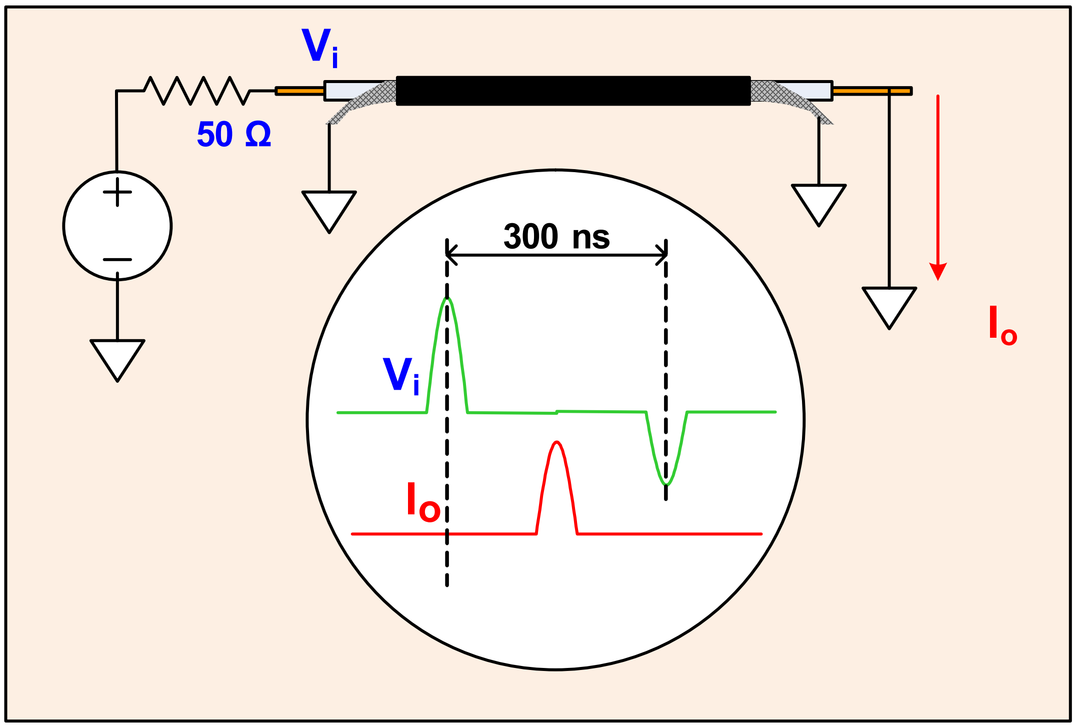
Input voltage, Vi, is a 1v pulse.
Output voltage, Vo, is a 0.8v pulse delayed 150 ns.
Output current, Iout, is a 16 ma pulse delayed by 150 ns.
There are no further pulses.



Case 2. Coax is shorted at the load, hence $\Gamma_2 = -1$

Vo	=	0
Iout	=	$(Y_2\ Vg)(F)$
Vi	=	$(Vg/2)(1-F^2)$

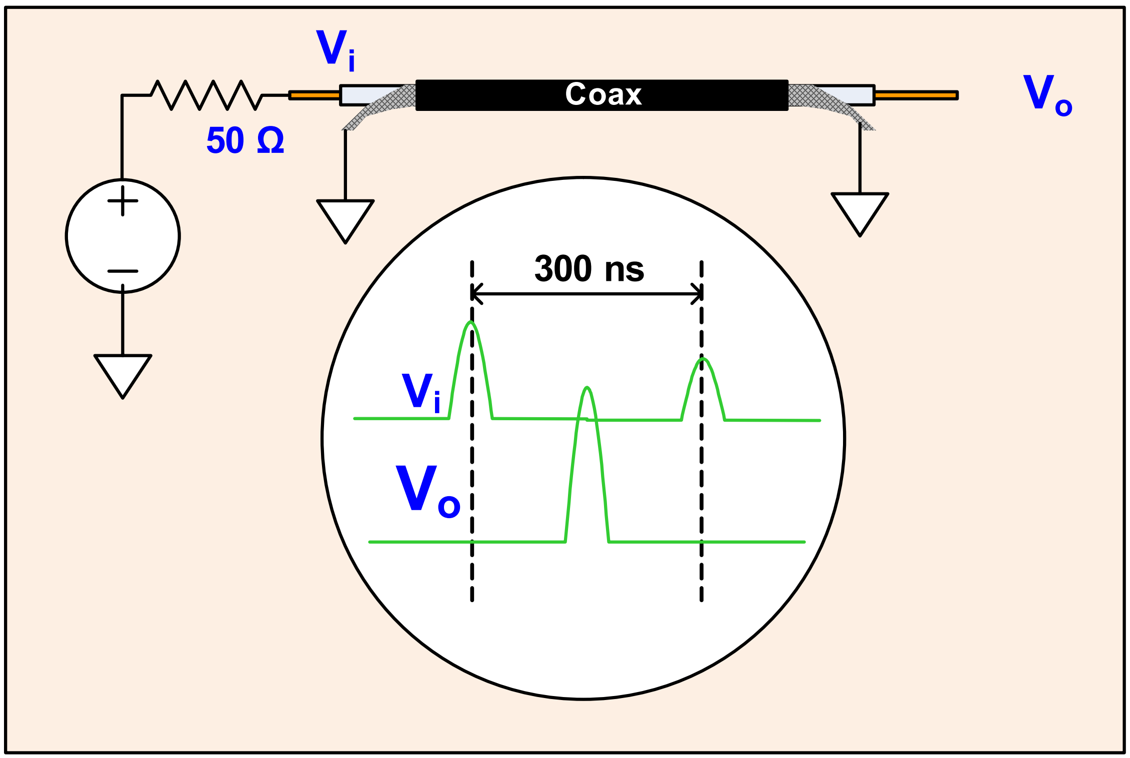
Output voltage, Vo, into a short circuit is zero.
Output current, Iout, is a 32 ma pulse delayed by 150 ns.
Input voltage, Vi, is a 1v pulse and then 300 ns later, a -0.64 v reflected pulse.
There are no further pulses.



Case 3. Coax is open at the load hence $\Gamma_2 = 1$

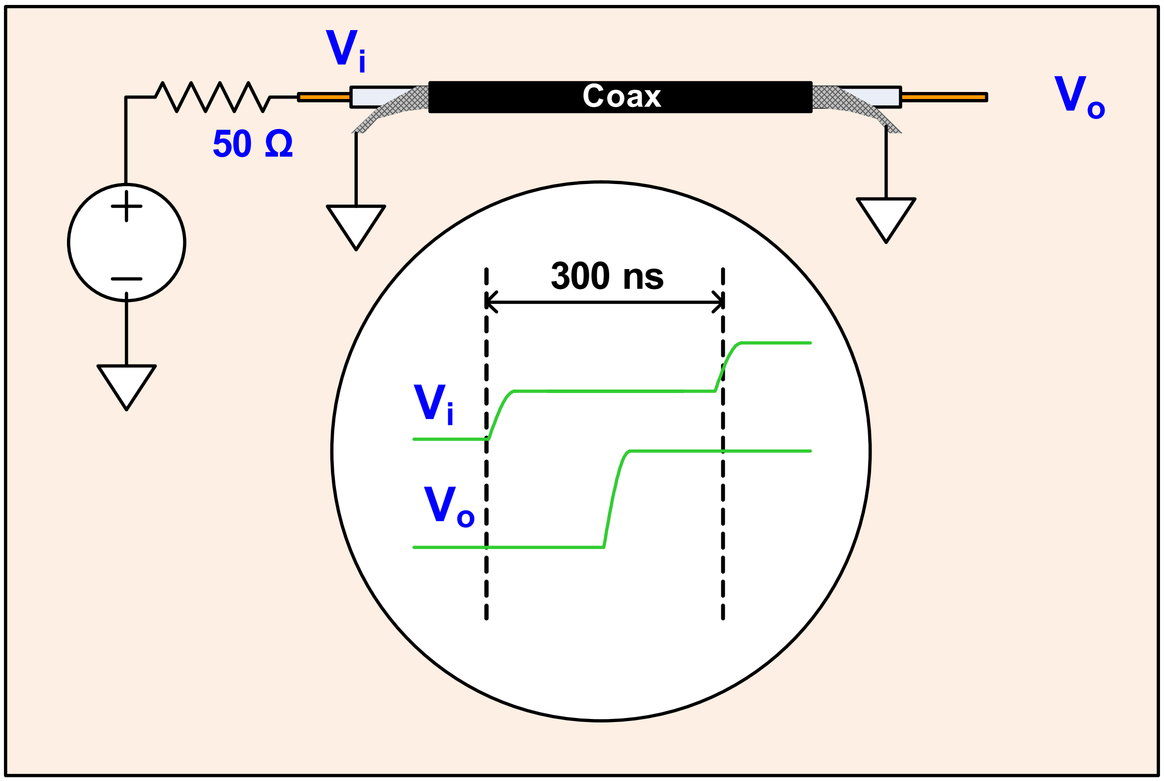
$$V_o = (V_g)(F)$$
$$I_{out} = 0$$
$$V_i = (V_g/2)(1+F^2)$$

Output voltage, Vo, is a 1.6v pulse delayed 150 ns.
Output current, Iout, into the open load is zero.
Input voltage, Vi, is a 1v pulse and then 300 ns later, a 0.64 v reflected pulse.
There are no further pulses.



Case 4. Coax is open at the load hence $\Gamma_2 = 1$ and the input is a step

Vo	=	(Vg) (F)
Iout	=	0
Vi	=	$(Vg/2)(1+F^2)$



A Closer Look at Telegrapher's Solutions

The behavior of Z and F have three distinct regions for ordinary transmission lines.

At high frequency, since we know the general form of the primary parameters, we can state that:

ωL_{ω}	>>	R_{ω}	
ωC_{ω}	>>	G_{ω}	
Z_{∞}	=	sqrt{ L_{∞} / C_{∞} }	a purely real constant
delay	=	$\beta/\boldsymbol{\omega}$ = sqrt { L_{∞} C_{∞} }	a constant
α	=	$R_{\omega}/(2Z_{\omega})$ $+Z_{\omega}G_{\omega}/2$	loss increases with ω because both R_{ω} and G_{ω} increase with frequency.

At intermediate frequencies:

$\boldsymbol{\omega} L_{\omega}$ << R_{ω}
$\boldsymbol{\omega} C_{\omega}$ >> G_{ω}
Z = sqrt {R_{dc}/ (jωC_{dc}) } which has a phase angle of -45° and increases inversely with ω
γ = sqrt { j$\boldsymbol{\omega} R_{dc} C_{dc}$ } = sqrt { $\boldsymbol{\omega} R_{dc} C_{dc}$ / 2 } + (j) sqrt { $\boldsymbol{\omega} R_{dc} C_{dc}$ / 2 } decreases with ω

At dc and very low frequencies:

$\omega L_{\omega} = 0$
$\omega C_{\omega} = 0$
Z_{dc} = sqrt { R_{dc} / G_{dc} } = a large real number
γ = sqrt {$R_{dc}G_{dc}$} = R_{dc}/Z_{dc} = loss = a small real number.

The DC Case

Does our model work at dc? It is not obvious from the unfamiliar forms that the solution takes. The characteristic impedance is very high. How, you ask, can you get any signal into it at all? The answer is that the line is mismatched at both ends and the reflections from the far end diminish the impedance at the near end and the reflections at the near end diminish the impedance at the far end. The two ends swap geometrically decreasing reflections that with a little smoothing look just like an RC charging curve. If you account for all the reflections in the end you wind up with exactly what you see on your osciloscope when you connect a dc source to a transmission line with a dc load.

The derivation and solution of the Telegrapher's Equations makes no assumption about the frequency of the signals and does not exclude dc. It ought to work. And in fact it does. In the zero frequency case, some terms disappear and others can be replaced by approximations in the form of simple monomials. At the end all that is left is what you would expect from dc circuit theory: if the line is not too long, the wire resistance and the load form a voltage divider and the dc voltage at the load is a less than at the source by just the right amount. And how long is too long? It depends on Gdc. If the line is long enough that the shunt conductance becomes significant. And what happens in that case? Nothing much. The telegraphers equations and solutions still work. The model still works. The only thing that doesn't work is dc circuit theory using lumped coponents.

How about SPICE. Does it work with this model at dc? The answer is yes for transient analysis, a qualified yes for AC analysis and a conditional yes for DC operating point analysis.

In the transient analysis, you put a step function (from zero to one for example) and you see the wave form settle down to a sustained dc level. The model definitely works in the transient analysis.

In the AC analysis, you cannot spefify 0 Hz, but you can make the low frequency as low as you want (even one cycle per year ~ 31 nHz). The model works in the AC analysis with the understanding that the frquency must be positive.

What about DC operating point? If the equations work and the model works and the transient and AC analysis work then the operating point analysis ought to work. Right? Yes it ought to work, but in the examples given in this article it does not. But we can make it work, by increasing the inductance of Lcon.

But, how do we feel about this? We feel disquieted. We are of the opinion that this guiding principal should hold: if the model is a sufficiently accurate model of the physical system, the simulator ought to produce accurate results. Thus, we should expend our effort on accurate modeling and not on appeasing the simulator. Once we get accurate simulation, we would be willing to tweak the model to run faster.

We are not bothered by documented, universal principals such as current sources are preferred to voltage sources or that Laplace functions should go to zero for infinite frequency. We are not bothered by undocumented, universal principals that we have discovered such as balanced circuits should be modeled by balanced models (which also is in accordance with our guiding principal of attempting to accurately model the physical system). We are a little bothered if we have to have a tweak such as Lcon that may need to be adjusted depending on rise times and transmission line length. But we are greatly bothered by having to change the value of the tweak for different types of analysis and especially if we have to change a reactance to get the dc operating point. Reactance should have no effect on the DC operating point.

So, what to do? We offer the following suggestion. If you can find a range of values for Lcon of say 100 to one where the outcome of a particular analysis does not change significantly, then set Lcon for the geometric mean of the range and use that value for that analysis. Certainly, when you do run any analysis you should try both larger and smaller values for Lcon.

Doing the Math for Zero Hertz

Let's examine the equation $Vo/Vi = (F)(1+\Gamma_2)/(1+F^2\Gamma_2)$ at dc.

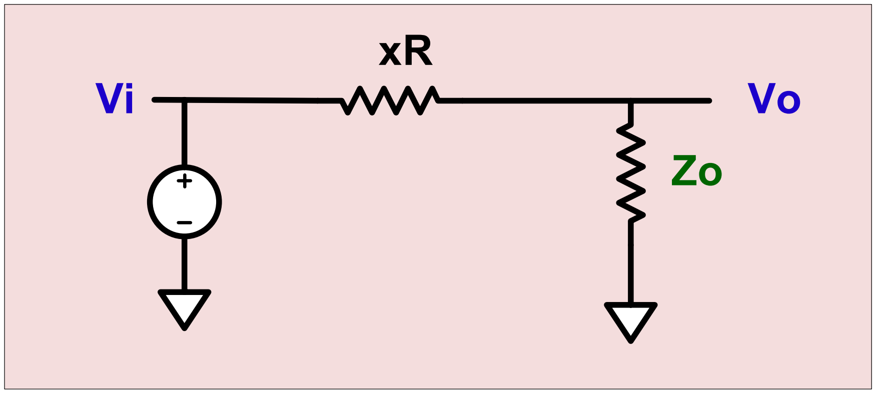


Figure 1. The equivalent circuit of a transmission line at DC.

Fig. 1 shows what we would expect from dc circuit theory. R is ohms per kft and x is the length in kft. So xR is just the total resistance of the wire between the input and the output.

$$
\begin{aligned}
\Gamma_2 &\equiv (Zo - Z2) / (Z2 + Zo) \\
Z1 &= \mathrm{Sqrt}\{ (R_\omega + sL_\omega) / (G_\omega + sC_\omega) \} . \\
Z2 &= Z1 \\
\gamma(s) &= \mathrm{Sqrt}\{ (R_\omega + sL_\omega) (G_\omega + sC_\omega) \} \\
F(s) &= e^{-x\gamma(s)}
\end{aligned}
$$

We will need some approximations. If m and n are very small numbers then:

$$
\begin{aligned}
1/(1+m) &\approx 1-m \\
e^{m} &\approx 1+m
\end{aligned}
$$

$(1+m)(1+n) \approx 1 + m + n$
$(1+m)^2 \approx 1 + 2m$

First, we set s=0 and ω=0 in the equations for Z1, Z2, F(s) and γ(s).

$$
\begin{aligned}
Z1 &= \text{Sqrt}\{ (R_{dc}) / (G_{dc}) \} \\
Z2 &= Z1 \\
\gamma(0) &= \text{Sqrt}\{ (R_{dc}) (G_{dc}) \} \\
F(0) &= e^{-x\gamma(0)}
\end{aligned}
$$

For the rest of this section we will supress the "dc" subscripts.

The term Sqrt { G / R } is a very small conductance per unit length that will show up often. We'll assign it the symbol symbol "ε".

$$\varepsilon = \text{Sqrt}\{ G / R \}$$

$$
\begin{aligned}
Z1 &= \text{Sqrt}\{ R/G \} \\
&= 1/\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_2 &= (Zo - Z2) / (Zo + Z2) \\
&= -(1 - \varepsilon Zo) / (1 + \varepsilon Zo) \\
&= -(1 - 2\varepsilon^2 Zo^2 + \varepsilon^2 Zo^2) / (1 - \varepsilon^2 Zo^2) \\
&\approx (-1 + 2\varepsilon Zo)
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= \text{Sqrt}\{ R\, G \} \\
&= (R) \text{Sqrt}\{ G/R \} \\
&= R\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
F &= e^{-x\gamma} \\
&= e^{-xR\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
& \approx && 1 - xR\varepsilon \\
\\
F^2 & \approx && (1 - xR\varepsilon)^2 \\
& \approx && 1 - 2xR\varepsilon \\
\\
F^2\Gamma_2 & \approx && (1 - 2xR\varepsilon)(-(1 - 2\varepsilon Zo)) \\
& \approx && -(1 - 2xR\varepsilon - 2\varepsilon Zo) \\
& = && -1 + 2xR\varepsilon + 2\varepsilon Zo \\
\\
Vo/Vi & = && (F)\,[\,1+\Gamma_2\,] / [\,1 + F^2\Gamma_2\,] \\
& \approx && (1 - xR\varepsilon)\,[1 + (-1 + 2\varepsilon Zo)\,] / [\,1 + (-1 + 2xR\varepsilon + 2\varepsilon Zo)\,] \\
& = && (1 - xR\varepsilon)\,(2\varepsilon Zo) / (2xR\varepsilon + 2\varepsilon Zo) \\
& = && (1 - xR\varepsilon)\,(Zo) / (xR + Zo) \\
& \approx && (Zo) / (xR + Zo)
\end{aligned}
$$

This says that the total wire resistance, xR, and the output load, Zo, form a voltage divider exactly as if the total wire resistance were a single lumped component. In other words, we get exactly what we would get from circuit theory. There is caveat. The term $xR\varepsilon$ may not be small because x, the length, is unlimited and could be very long. The telegrapher's equations still work. The two port model still works. But the distributed shunt conductance has become large enough that the approximations are not sufficiently accurate to apply the theory of circuits constructed from lumped elements.