CHAPTER 9

TUNED CIRCUITS

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SECTION 1 : INTRODUCTION

When a violin is tuned, the tensions of its strings are adjusted to permit vibration at particular frequencies. In radio, when an arrangement of L, C, and R responds to particular frequencies, it is called a " tuned " circuit.

In principle, the tuned circuit is similar to a pendulum or violin string, tuning fork, etc.—it has the property of storing energy in an oscillating (vibrating) state, regularly changing from kinetic form (magnetic field when current flows through the coil) to potential form (electric field, when the condenser is charged) and back again at a frequency called the **natural resonant frequency.**

FIG. 9.1

In Fig. 9.1 let the condenser C be charged. It will discharge its energy through the inductance L, causing the current to increase all the while, until it reaches the maximum when there is no potential across C. At that instant the energy is all magnetic, and the current continues, fed by the magnetic field, to build a voltage of reversed polarity across C. When all the energy has been transferred from L to C, the voltage across C has its original value, but is reversed in sign, and the current is diminished to zero. The process then reverses, and repeats itself indefinitely, cycle after cycle.

SECTION 2 : DAMPED OSCILLATIONS

Were there no loss of energy, each cycle would be the same, but this is never the case in practice, as the coil must have resistance. The rate at which the energy decreases is proportional to the energy (remaining) in the circuit.

For a detailed account of damped oscillations, as these periodic changes of decreasing amplitude are called, reference should be made to standard textbooks relating to the theory of radio circuits. A list of several such books is given in the accompanying bibliography.

Here we consider briefly the circuit shown in Fig. 9.2a. Let I_o be the maximum value of the current during a given cycle of the oscillation, and let the time t be measured from the instant of this maximum ; then the value of the current i at any subsequent time is given by the equation

$$i = I_o \epsilon^{-\alpha t} \cos \omega_n t, \tag{1}$$

where $\alpha = r/2L = $ damping factor,

and $\omega_n = \sqrt{\dfrac{1}{LC} - \dfrac{r^2}{4L^2}}$.

The quantity $f_n = \omega_n/2\pi$ is called the **natural resonant frequency**. In the above formulae L is in henrys, C in farads, r in ohms, ω_n in radians per second and f_n in cycles per second.

The **resonant frequency** f_o (corresponding to $r = 0$) of the circuit is defined by

$$2\pi f_o = \omega_o = \sqrt{1/LC} \tag{2}$$

The ratio of the natural resonant frequency to the resonant frequency is

$$f_n/f_o = \sqrt{1 - 1/(4Q^2)} \tag{3}$$

where $Q = \dfrac{\omega_o L}{r} = \dfrac{2\pi f_o L}{r}$

$$= \frac{\text{reactance of the coil at resonant frequency}}{\text{coil resistance}}.$$

This relationship shows that in practice there is little difference between these two frequencies. Q must, for example, be less than four to make f_n differ by 1% from f_o. Q normally exceeds fifty, for which value the two frequencies differ by about one part in twenty thousand.

If now, in the equation for the current, t is increased by an amount $2\pi/\omega_n$, the period of one cycle, we arrive at the corresponding point in the next cycle. Let i' denote the current at this point, so that

$$i' = I_o \epsilon^{-\alpha(t + 2\pi/\omega_n)} \cdot \cos \omega_n(t + 2\pi/\omega_n)$$

$$= i\epsilon^{-2\pi\alpha/\omega_n}$$

or, $i'/i = \epsilon^{-\pi r/\omega_n L}$, since $\alpha = r/2L$ (4)

This gives the ratio of the amplitude of one cycle to that immediately preceding.

Logarithmic decrement, δ, is defined by

$$\delta = \pi r/\omega_n L = \log_\epsilon (i/i') \tag{5}$$

and is thus the naperian logarithm of the ratio of the amplitudes of two successive cycles.

See also Sect. 11 Summary of Formulae.

SECTION 3 : SERIES RESONANCE

The **series impedance** z of the circuit in Fig. 9.2a at any frequency $(f = \omega/2\pi)$ is given by the expression

$$z = \sqrt{r^2 + [\omega L - (1/\omega C)]^2} \tag{6}$$

Thus if an alternating voltage of frequency f be applied in series with the circuit, when the value of r is fixed, we see that z is least, and hence the current reaches its maximum, when $\omega L - 1/\omega C = 0$, i.e. when $\omega = 1/\sqrt{LC} = \omega_o$, the resonant frequency.

It is perhaps surprising that the frequency for maximum current is independent of the circuit resistance r and that maximum current does not occur when the frequency of the applied voltage is equal to the natural frequency f_n. The state of maximum current flow is called **series resonance**.

If E be the r.m.s. value of the applied alternating voltage, the r.m.s. value I of the current produced in the circuit is clearly

$$I = \frac{E}{z} = \frac{E}{\sqrt{r^2 + [\omega L - (1/\omega C)]^2}} \tag{7}$$

$$= E/r, \text{ when } \omega = 1/\sqrt{LC} = \omega_o \tag{8}$$

The voltages across the several parts of the circuit under this condition of series resonance are (Fig. 9.2a) :

$$rI = E \text{ across the resistance} \tag{9}$$
$$\omega_o LI \text{ across the inductance} \tag{10}$$
and $$- (1/\omega_o C)I \text{ across the capacitance.} \tag{11}$$

The voltages across the inductance and capacitance are equal and opposite in sign, thus cancelling each other, and usually they are large compared with the voltage across the resistance. The voltage across the inductance may be expressed as $(\omega_o L/r)E$, and therefore its ratio to the voltage applied in series with the circuit is $\omega_o L/r$. **This ratio, usually denoted by Q,** is the ratio of the reactance of the coil at resonance to the resistance in series with it, and is called the **magnification factor** or **quality factor**. Thus,

$$Q = \frac{\omega_o L}{r} = \frac{1}{r}\sqrt{\frac{L}{C}} = \frac{1}{\omega_o C r} \tag{12}$$

Q may also be called the **energy factor** and is then defined by

$$Q = 2\pi \frac{\text{peak energy storage}}{\text{energy dissipated per cycle}}$$

$$= \omega \frac{\text{peak energy storage}}{\text{average power loss}}.$$

That this is equivalent to $\omega_o L/r$ is shown by

$$Q = \omega_o \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2 r} = \frac{\omega_o L}{r}.$$
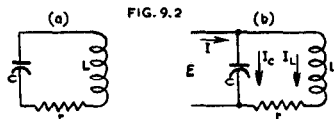

FIG. 9.2

It is shown in Chapter 4 Sect. 5(vi) that the **power factor** is approximately equal to $1/Q$; when Q is greater than 7 the error is less than 1%.

See also Sect. 11 Summary of Formulae.

The general treatment of series circuits with L, C and R is given in Chapter 4 Sect. 6(ii).

SECTION 4 : PARALLEL RESONANCE

Let us now examine the result of applying an alternating voltage E across the condenser C in the circuit shown in Fig. 9.2b. The current divides between the two branches, and we find that the (r.m.s.) value of the current in L and r is given by

$$I_L = \frac{E}{\sqrt{r^2 + \omega^2 L^2}} \tag{13}$$

and the r.m.s. value of the current in C by

$$I_c = \omega C E \tag{14}$$

Adding the two currents, with due regard to their phase relation, the r.m.s. value of the total current I is found to be

$$I = E \sqrt{\left(\omega C - \frac{\omega L}{r^2 + \omega^2 L^2}\right)^2 + \left(\frac{r}{r^2 + \omega^2 L^2}\right)^2} \tag{15}$$

When $\omega C = \omega L/(r^2 + \omega^2 L^2)$, the total current I is in phase with the applied voltage E, and has its minimum value

$$I = E \frac{r}{r^2 + \omega^2 L^2} \tag{16}$$

This condition is termed " parallel resonance* ", as distinct from the " series resonance " considered earlier. At resonance, the currents in the condenser and coil are large compared with the current in the external circuit and they are very nearly opposite in phase.

The value of ω at which parallel resonance occurs can be determined from the equation

$$\omega_0 = \frac{1}{\sqrt{LC}} \cdot \frac{1}{\sqrt{1 + (r^2/\omega_o^2 L^2)}} \tag{17}$$

It was pointed out in the previous section that $\omega L/r$, the ratio of the reactance of the coil to the resistance in series with it, is usually greater than 50. Thus, with an error of about one part in 5000 at $Q = \omega L/r = 50$ and correspondingly smaller errors at larger values of Q, we find that the parallel resonant frequency f_o is given by

$$\omega_o = 2\pi f_o \approx 1/\sqrt{LC}.$$

This is the same result, to the accuracy indicated above, as that obtained in the series resonance case.

With a similarly small degree of error we may write the following simple relations for the **currents at resonance.** The current in the external circuit (Fig. 9.2b) is given by

$$I \approx E \frac{r}{\omega_o^2 L^2} \approx \frac{E.Cr}{L} \approx E.\omega_o^2 C^2 r,$$

$$\approx \frac{E}{\omega_o L} \cdot \frac{1}{Q} \approx \frac{E.\omega_o C}{Q} \tag{18}$$

where $Q = \omega_o L/r$ as before. **The current in the inductance and resistance** is very closely equal in value to that in the capacitance,

$$I_L \approx - I_c \approx E/\omega_o L \approx - \omega_o C E \tag{19}$$

and **is Q times larger than the current in the external circuit**

The formulae given so far hold only when the condenser loss is negligible. In order to generalize our expressions in a simple manner let us first consider the two circuits shown in Fig. 9.3a and Fig. 9.3b. At (parallel) resonance for Fig. 9.3a we have found that the current I in the external circuit is

$$I \approx ECr/L \approx E/Q\omega_o L \text{ etc. };$$

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*Parallel resonance may be defined either as
  (a) the frequency at which the parallel impedance of the circuit is a maximum, or
  (b) the frequency at which the equivalent reactance of the complete parallel circuit becomes zero
    (i.e. when the impedance has unity power factor and acts as though it were a pure resistance at
    the resonant frequency). This can also be expressed by saying that the parallel circuit has
    zero susceptance at the resonant frequency.
  For further details see Chapter 4 Sect. 6(iii) and (iv).
Definition (b) is used in this chapter.

and for Fig. 9.3b, to the same approximation, it is evident that
$$I \approx E/R_e.$$
These two circuits are equivalent at resonance provided we set
$$R_e = L/Cr = Q\omega_oL = Q/\omega_oC = Q\sqrt{L/C} \qquad (20)$$
Note that $Q = R_e\sqrt{C/L}$, and that R_e is the parallel resistance (across C) equivalent to the series coil resistance r.

It has been shown that the circuit Fig. 9.3a with a condenser C having no losses and an inductance L having series coil resistance r may be replaced by the equivalent circuit Fig. 9.3b having an ideal tuned circuit LC, without losses, shunted by the resistor R_e having a value given by equation (20). It is obvious that the impedance of the parallel combination LCR_e in Fig. 9.3b at resonance is R_e, this being the "resonant impedance" of the circuit. At frequencies other than the resonant frequency, the impedance will be less than the "resonant impedance."

The values of Q in terms of series coil resistance r and equivalent parallel resistance R_e are grouped below for convenience.

In terms of r : $Q = \dfrac{\omega_oL}{r} = \dfrac{1}{\omega_oCr} = \dfrac{1}{r}\sqrt{\dfrac{L}{C}}$

In terms of R_e : $Q = \dfrac{R_e}{\omega_oL} = \omega_oCR_e = R_e\sqrt{\dfrac{C}{L}}$

We can now consider the important practical case of the circuit shown in Fig. 9.4, in which a resistance R appears in shunt with the condenser C. R represents the effect of all insulation losses in condenser, coil, wiring, switches and valves, together with the plate or input resistance of the valves.
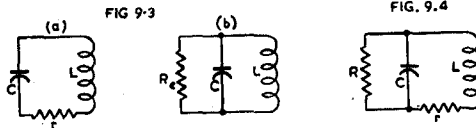

Fig. 9.3(a).   *Parallel resonance with series loss resistance.*
Fig. 9.3(b).   *Parallel resonance with parallel loss resistance.*
Fig. 9.4.   *Parallel resonance with both series and parallel loss resistances.*

In the present case we have R in parallel with our equivalent parallel coil resistance R_e of Fig. 9.3b. The resultant parallel resistance at resonance, which we will call the **resonant impedance** is, of course,
$$R_D = \frac{1}{1/R + Cr/L} \qquad (21)$$
Therefore, the **resultant value of Q is**
$$Q = \sqrt{\frac{C}{L}} \cdot R_D = \frac{1}{(1/R)\sqrt{L/C} + r\sqrt{C/L}} = \frac{1}{(\omega_oL/R) + (r/\omega_oL)} \qquad (22)$$

Note that at the resonant frequency the expression $\sqrt{L/C}$ is equal to the reactance of the inductance and also that of the condenser, i.e.
$$\omega_oL = \sqrt{L/C} = 1/\omega_oC. \qquad (23)$$
See also Sect. 11 Summary of Formulae.

SECTION 5: GENERAL CASE OF SERIES RESONANCE

Fig. 9.5 shows the general type of series resonant circuit. It is often convenient to express the effect of the two resistances at resonance as a resultant equivalent series resistance, r' say. In the circuit of Fig. 9.3 we saw that the effect of a resistance r in series with L was equivalent at resonance to that of a resistance L/Cr shunted across C. By similar reasoning it may be shown in the present case (Fig. 9.5) that
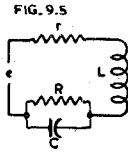
FIG. 9.5.

Fig. 9.5. *Series resonance with losses in inductive and capacitive elements.*

the effect of R at resonance is equivalent to that of a resistance of value L/CR in series with L. The resultant equivalent series resistance r' is thus equal to $r + (LC/R)$, and the resultant value of Q is $\omega_o L/r'$.

See also Sect. 11 Summary of Formulae.

SECTION 6: SELECTIVITY AND GAIN

(i) Single tuned circuit (ii) Coupled circuits—tuned secondary (iii) Coupled circuits—tuned primary, tuned secondary (iv) Coupled circuits of equal Q (v) Coupled circuits of unequal Q.

(i) Single tuned circuit

The currents and voltages, and hence the gain, of single tuned circuits at resonance are determined by the equivalent series resistance (r' as defined in Sect. 5) and the resonant impedance (R_D as defined in eqn. 21).

At frequencies other than the resonant frequency, the reactances of the coil and the condenser no longer balance. In the series circuit the resistance r' becomes an impedance z which is greater than r'. In the parallel circuit the resonant impedance R_D becomes an impedance Z which is less than R_D.

The appropriate ratios of these quantities determine the **selectivity** or response of the circuit, and they are related to Q and f by the following expression :

$$\frac{A_o}{A} = \frac{z}{r'} = \frac{R_D}{Z} = \sqrt{1 + Q^2\left(\frac{f}{f_o} - \frac{f_o}{f}\right)^2} \qquad (24)$$

where A_o is the voltage gain at the resonant frequency f_o and A the gain at frequency f.

Note that A_o/A is the ratio of current at resonance to that at frequency f in the series case, and the ratio of **total** current at frequency f to that at the resonant frequency in the parallel case (see below).

The phase angle between the applied voltage and the total current is such that

$$\tan \phi = \pm\, Q\left(\frac{f}{f_o} - \frac{f_o}{f}\right) \qquad (25)$$

the positive sign pertaining to the series case and the negative sign to the parallel case.

The equation (24) leads to a simple method for determining Q from the response curve. We see that when $Q[(f/f_o) - (f_o/f)] = \pm 1$ the total current will be decreased by the factor $\sqrt{2}$ in the series case (i.e. decreased to 70.7% of the resonance value), and increased by the same factor in the parallel case. It will be observed that at either of the frequencies satisfying this condition the phase angle ϕ is numerically equal to $45°$, as $\tan \phi = \pm 1$, and that the resistance r' or R_D is equal to the reactance.

The condition
$$Q(f/f_o - f_o/f) = \pm 1 \tag{26}$$
may be written
$$f/f_o - f_o/f = \pm 1/Q ; \tag{27}$$
and for values of Q not too low (> 50, say) we have, very closely,
$$2\Delta f_o/f_o \approx \pm 1/Q \tag{28}$$
where $\Delta f_o = f - f_o$.

Thus $\Delta f_o \approx \pm \dfrac{f_o}{2Q}$ \hfill (29)

and the current is decreased, or increased, by the factor $\sqrt{2}$ at two frequencies f_1 and f_2, one on each side of the resonant frequency f_o, determined by
$$f_1 = f_o - \frac{f_o}{2Q} ; f_2 = f_o + \frac{f_o}{2Q} \tag{30}$$
Hence, $Q \approx f_o/(f_2 - f_1)$ \hfill (31)
The frequencies f_o, f_1 and f_2 may be found experimentally, and hence Q may be calculated.

At frequencies very different from f_o, so that A_o/A is greater than about 10, the equation giving the response is very approximately
$$A_o/A \approx Q(f/f_o - f_o/f). \tag{32}$$
Also, the expression for the phase angle may be written in the alternative forms
$$\tan \phi = \pm Q(f/f_o - f_o/f) \qquad \text{from equation (25)}$$
$$= \pm Q \frac{\Delta f_o}{f_o} \cdot \frac{2 + (\Delta f_o/f_o)}{1 + (\Delta f_o/f_o)} \tag{33}$$
Under these conditions we see that **in the series case**

(a) Across the coil :
$$\frac{\text{Voltage at resonance}}{\text{Actual voltage when well off resonance}} = \frac{L\omega_o i_o}{L\omega i} = \frac{Lf_o i_o}{Lfi} = Q\left(1 - \frac{f_o^2}{f^2}\right) \tag{34}$$
and (b) Across the condenser :
$$\frac{\text{Voltage at resonance}}{\text{Actual voltage when well off resonance}} = \frac{C\omega i_o}{C\omega_o i} = Q\left(\frac{f^2}{f_o^2} - 1\right) \tag{35}$$
Similarly, **in the parallel case**, we have
$$\frac{\text{Impedance at resonance}}{\text{Impedance at frequencies well off resonance}} = Q\left(\frac{f}{f_o} - \frac{f_o}{f}\right) \tag{36}$$
from which it can be shown that

(a) In the coil :
$$\frac{\text{Current at resonance}}{\text{Current at frequency } f} = \frac{f}{f_o} \tag{37}$$
and (b) in the condenser :
$$\frac{\text{Current at resonance}}{\text{Current at frequency } f} = \frac{f_o}{f} \tag{38}$$

(ii) Coupled circuits—tuned secondary

We consider briefly now two examples of coupled circuits. The first example, shown in Fig. 9.6 illustrates a typical case of a **high frequency transformer with tuned secondary** in a radio receiver.

The symbols to be used are set out below :

g_m = mutual conductance of the valve in mhos,

r_p = plate resistance of the valve in ohms,

L_1 = primary inductance in henrys,

L_2 = secondary inductance in henrys,

M = mutual inductance between L_1 and L_2 in henrys,

k = $M/\sqrt{L_1 L_2}$ = coupling factor,

Q_2 = $L_2 \omega_o/r = 1/rC\omega_o$,

ω_o = $2\pi \times$ resonant frequency of secondary in cycles per second,

$R_D = \omega_o L_2 Q_2 =$ resonant impedance of secondary in ohms,

$e_i =$ input voltage,

$e_o =$ output voltage,

and $A =$ stage voltage gain (amplification).

When the secondary is tuned, its impedance (at ω_o) is simply r, and it reflects into the primary a resistance equal to $M^2 \omega_o^2/r$. The primary signal current I_p is, therefore,

$$I_p = \frac{g_m \cdot r_p \cdot e_i}{\sqrt{X_p^2 + (r_p + M^2\omega_o^2/r)^2}} \tag{39}$$

where X_p is the reactance of the primary.

When the conditions are such that the primary reactance can be neglected, we have

$$I_p \approx g_m \cdot r_p \cdot e_i/(r_p + M^2\omega_o^2/r) \tag{40}$$

It follows that the secondary current I_s is given by

$$I_s = \frac{M\omega_o I_p}{r} \approx \frac{g_m \cdot e_i}{(r/M\omega_o) + (M\omega_o/r_p)} \tag{41}$$

Hence the induced voltage across L_2 (and C), that is e_o, is

$$e_o = \frac{g_m \cdot e_i \cdot L_2\omega_o}{(r/M\omega_o) + (M\omega_o/r_p)} \tag{42}$$

or $$\frac{e_o}{e_i} = A_o = \frac{g_m}{(1/M\omega_o Q_2) + (M/r_p L_2)} \tag{43}$$

$$= \frac{g_m k R_D \sqrt{L_1/L_2}}{1 + k^2(R_D/r_p) \cdot (L_1/L_2)} \tag{44}$$

When r_p is very much greater than $\omega_o^2 M^2/r[= k^2 R_D(L_1/L_2)]$ we have simply

$$I_p \approx g_m \cdot e_i, \tag{45}$$

$$I_s \approx g_m \cdot e_i \cdot M\omega_o/r, \tag{46}$$

and $e_o/e_i = A_o \approx g_m \omega_o M Q_2 = g_m \cdot k R_D \sqrt{L_1/L_2}$ ... (47)

If the inductances L_1 and L_2 have the same ratio of diameter to length, or form factor, and the turns are N_1 and N_2 respectively, then $\sqrt{L_1/L_2}$ in the above formulae may be replaced by N_1/N_2. The plate resistance r_p in parallel with the primary is reflected as a series resistance $M^2\omega_o^2/r_p$ into the secondary. If the value of this reflected resistance is greater than say 5% of r, its effect should be taken into account when computing the selectivity of the secondary. This selectivity, then, is determined by means of the formula

$$z/r' = \sqrt{1 + Q_2'^2[f/f_o - f_o/f]^2} \tag{48}$$

where $r' = r + M^2\omega_o^2/r_p$,

and $Q_2' = \dfrac{\omega_o L_2}{r'} = \dfrac{1}{(1/Q_2) + (k^2 L_1 \omega_o/r_p)} \cdot$

Note that Q_2' and r' must not be used when calculating the gain A_o.
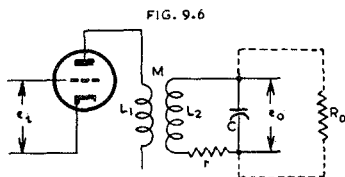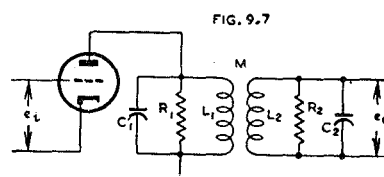
FIG. 9.6   FIG. 9.7

Fig. 9.6. Amplifier stage using a high frequency transformer with a tuned secondary.
Fig. 9.7. Amplifier stage using a double tuned high frequency transformer.

(iii) Coupled circuits—tuned primary, tuned secondary

The second example of coupled circuits is shown in Fig. 9.7. This is a typical high frequency transformer with tuned primary and tuned secondary. Intermediate frequency transformers in super-heterodyne receivers are usually of this type. Very thorough discussions of such transformers have been given from the theoretical point

of view by Aiken (Ref. C2). For the design procedure see Chapter 26 Sect. 4. In this case the resistance in parallel with the primary at resonance is

$L_1/C_1 r_p + L_1/C_1 R_1 = L_1/C_1 R'$, where $1/R' = 1/r_p + 1/R_1$;

and the resistance of the secondary is $L_2/C_2 R_2$, so that the resistance reflected into the primary is $\omega_o{}^2 M^2 C_2 R_2/L_2$. The primary current I_p is, therefore,

$$I_p = \frac{g_m e_i}{\omega_o C_1} \cdot \frac{1}{L_1/C_1 R' + \omega_o{}^2 M^2 C_2 R_2/L_2} \tag{49}$$

The secondary current I_s can readily be shown to be

$$I_s = (M\omega_o C_2 R_2/L_2) I_p, \tag{50}$$

and the induced voltage e_o across L_2 and C_2 to be

$$e_o = L_2 \omega_o I_s = (M C_2 R_2 \omega_o{}^2) I_p.$$

Therefore the voltage gain at the resonant frequency, A_o, is given by

$$\frac{e_o}{e_i} = A_o = \frac{g_m M \omega_o}{L_1/R' R_2 C_2 + \omega_o{}^2 M^2 C_1/L_2} \cdot \tag{51}$$

This relation for the gain may also be expressed as

$$A_o = \frac{g_m k \omega_o \sqrt{L_1 L_2}}{k^2 + 1/Q' Q_2} \tag{52}$$

or, $$A_o = \frac{g_m \sqrt{R' R_2}}{k\sqrt{Q' Q_2} + 1/(k\sqrt{Q' Q_2}}, \tag{53}$$

where, as before, $k = M/\sqrt{L_1 L_2}$, and Q', Q_2 have their usual meaning, i.e. $R'/L_1 \omega_o$ and $R_2/L_2 \omega_o$ respectively.

The expression for calculating the selectivity is lengthy and complicated and a graphical treatment described later is preferable (Sect. 7).

As k is increased from low values, the gain increases until $k\sqrt{Q' Q_2} = 1$, after which it decreases. This value of $k = 1/\sqrt{Q' Q_2}$ is known as the **critical coupling factor** (k_c).

(iv) Coupled circuits of equal Q

When the primary and secondary circuits are identical, and the coupling factor is equal to k_c, and the plate and grid return resistances are very high, we see that the voltage gain obtained is exactly half that with a single tuned circuit. The critical coupling factor in this case, of course, is

$k_c = 1/Q$ where $Q = Q' = Q_2$.

For values of k less than k_c, the response curve (gain versus frequency) has a single maximum at f_o, the resonant frequency of each of the circuits. When k exceeds the critical value, however, the amplification curve becomes double-humped, i.e. there are two frequencies of maximum response, and these are separated by equal amounts above and below f_o. The distance between these two peaks increases with k, and very approximately we find that

$$(f_2 - f_1)/f_o \approx \sqrt{k^2 - 1/Q^2} \tag{54}$$

$$\approx \sqrt{k^2 - k_c{}^2}, \tag{55}$$

when f_1 and f_2 are the frequencies for maximum response, i.e. $(f_2 - f_1)$ is **the band width between peaks**. The amplitude of these two peaks is substantially the same as the maximum possible gain $g_m R/2$, where $R = R_1 = R_2$.

Frequently, an approximate formula for band-width is used:

$$(f_2 - f_1)/f_o \approx k \tag{56}$$

While k largely determines the band-width, the depth of the valley at f_o, and hence the uniformity of the response in the pass band of frequencies, is determined by the relation of Q to k. For a constant value of k (above critical coupling) the dip becomes more pronounced as Q is increased, while the frequency separation between peaks becomes greater; conversely as Q is decreased the dip becomes less pronounced and the frequency separation between peaks becomes less. The ratio of the response at f_o to that at the two peaks is found to be $2.b/(1 + b^2)$, where $b = k/k_c$.

A value for k in the order of 1.5 times critical, i.e. $kQ = 1.5$, is often used for i-f amplifiers requiring band pass characteristics. However, the exact value chosen for kQ depends of course, on the bandwidth requirements.

Further points on the resonance curve can be obtained from the result that the frequency band width between the points on either flank of the resonance curve, at which the response is equal to the minimum in the " valley " between the two peaks, is $\sqrt{2}$ times the peak separation. It can be shown also that, in general, the gain at any frequency f is given by

$$\frac{A_o}{A} = \sqrt{\left[1 - \frac{Q^2 Y^2}{1 + k^2 Q^2}\right]^2 + \left[\frac{2QY}{1 + k^2 Q^2}\right]^2} \tag{57}$$

where $Y = f/f_o - f_o/f$.

This expression may well be solved graphically according to a procedure developed by Beatty (Ref. C7) ; this procedure will be described below.

(v) Coupled circuits of unequal Q

In the general case where Q' and Q_2 are unequal, the two peaks of maximum response do not appear immediately k exceeds the critical value k_c. The value of k at which the two peaks just appear has been defined as the **transitional coupling factor** by Aiken (Ref. C2). The value of this coupling factor k_t is

$$k_t = \sqrt{\tfrac{1}{2}(1/Q'^2 + 1/Q_2^2)}. \tag{58}$$

The band width between peaks is found to be

$$(f_2 - f_1)/f_o = \sqrt{k^2 - k_t^2} \tag{59}$$

This useful result is discussed in an editorial by G. W. O. Howe (Ref. C5). Further it has been shown that here, as in the symmetrical case, the band width between the points on the flanks, level with the minimum response in the " valley " between the peaks, is $\sqrt{2}$ times the peak separation.

In this case, too, the selectivity curve remains symmetrical as k increases (above k_t). The amplitude of the peaks decreases, however, as k increases, and also as the ratio of R_1/R_2 (or R_2/R_1) increases. Aiken (Ref. C2) gives selectivity curves for the three cases (i) $R_1 = 10R_2$, (ii) $R_1 = 50R_2$ and (iii) $R_1 = 200R_2$ with $L_1 = L_2$ and $C_1 = C_2$. Some idea of the magnitude of this decrease in peak amplitude may be obtained from the following figures taken from Aiken's curves :

$R_1/R_2 = 10$		$R_1/R_2 = 50$		$R_1/R_2 = 200$	
k/k_c	A/A_{opt}	k/k_c	A/A_{opt}	k/k_c	A/A_{opt}
3	0.67	7	0.33	15	0.17
6	0.62	15	0.28	20	0.15

In this table A_{opt} is the optimum value of the gain, i.e. the gain at resonant frequency with critical coupling ($= g_m \sqrt{R_1 R_2}/2$).

The gain at f_o when $k > k_t$ is given by $[2b/(1 + b^2)]A_{opt}$, where $b = k/k_c$, as in the case of equal Q's ; as the gain at the peaks is less than A_{opt} however, the response curve is flatter in the present case.

See also Sect. 11 Summary of Formulae.

SECTION 7 : SELECTIVITY—GRAPHICAL METHODS

(i) Single tuned circuit   (ii) Two identical coupled tuned circuits.

(i) Single tuned circuit

A single tuned circuit in the plate load of a valve has the well-known frequency response shown in Fig. 9.8. At the resonant frequency, where the reactance is zero

there occurs the maximum value of the response, and the gain falls away on both sides. This curve may be computed from eqn. (24), namely

$$A_o/A = \sqrt{1 + Q^2 Y^2},$$

and $\tan \phi = \pm QY$,   from eqn. (25)

where $Y = (f/f_o - f_o/f)$.

These formulae, however, lend themselves to a simple graphical treatment as indicated in Fig. 9.9.

The ratio of the gain A_o at resonance to the gain A at any other frequency may be plotted as a vector quantity, OP in Fig. 9.9, having both magnitude and phase. At resonance, when the frequency is f_o, it becomes OP_o in Fig. 9.9, where OP_o is of unit length since A_o/A is then equal to unity. At any other frequency f, the ratio A_o/A is then given by OP where the point P is fixed by the relation

length $P_o P = Q(f/f_o - f_o/f)$.

The phase angle is the angle $P_o OP$.

Near resonance, when f is nearly equal to f_o, this may be approximated by

length $P_o P \approx 2Q \Delta f/f_o$ where $\Delta f = f - f_o$

so that Δf measures the amount of detuning. Thus, near resonance, the length $P_o P$ is nearly proportional to the amount of detuning.
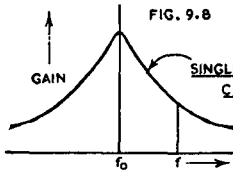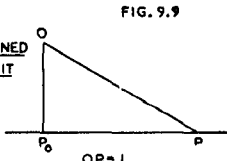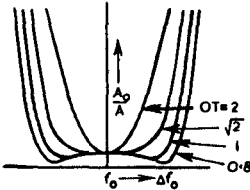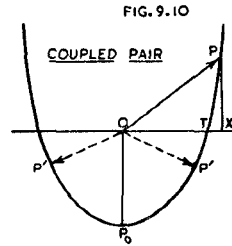

FIG. 9.8   FIG. 9.9   FIG. 9.10

FIG. 9.11

SINGLE TUNED CIRCUIT

COUPLED PAIR

WHERE :—

A_o = GAIN AT RESONANCE
A = GAIN AT f
f_o = RESONANT FREQUENCY
$Q = \dfrac{2\pi f_o L}{R}$
$Y = \left[\dfrac{f}{f_o} - \dfrac{f_o}{f}\right]$
K = COEFFICIENT OF COUPLING

$OP_o = 1$
$OP = \dfrac{A_o}{A}$
$P_o P = Q\left[\dfrac{f}{f_o} - \dfrac{f_o}{f}\right]$

$OP_o = 1$
$OP = \dfrac{A_o}{A}$
$OX = \dfrac{2QY}{1+K^2 Q^2}$
$XP = 1 - \dfrac{Q^2 Y^2}{1+K^2 Q^2}$
$OT = \dfrac{2}{\sqrt{1+K^2 Q^2}}$

$OT = 2$
$\sqrt{2}$
1
0.8

(ii) Two identical coupled tuned circuits

It is found that when two identical tuned circuits are coupled, either by some common reactance in the circuit or by mutual inductance, the locus of the point P is the parabola

$$x^2 = (y + 1)4/(1 + k^2 Q^2).$$

This parabola is shown in Fig. 9.10 where OP_o again represents A_o. To use this graph to determine the gain, we first compute the quantity $OX = 2QY/(1 + k^2 Q^2)$, then from X draw a line perpendicular to OX to cut the curve at P. The line OP represents the gain (A_o/A) while $P_o OP$ is the phase angle.

The form of the parabola depends upon the magnitude of kQ. It is found that when $OT < \sqrt{2}$, corresponding to $kQ > 1$, there are two frequencies of maximum gain as shown by the two vectors marked OP' in Fig. 9.10. When the attenuation is plotted against Δf_o as in Fig. 9.11, it becomes clear that a much flatter top may be obtained by using coupled pairs of circuits than by using single tuned circuits. Fig. 9.11 serves also to show the variation in band width with variations of OT (i.e. changes of kQ). It will be seen that the shape of the skirt of the curve is practically independent of the value of kQ.

See also Sect. 11 Summary of Formulae.

SECTION 8 : COUPLING OF CIRCUITS

(*i*) *Mutual inductive coupling* (*ii*) *Miscellaneous methods of coupling* (*iii*) *Complex coupling.*

(i) Mutual inductive coupling

As already emphasized, when the mutual inductive coupling between two tuned circuits is increased above a critical value, k_t, two peaks appear in the response curve, symmetrically situated with regard to the resonant frequency f_o. No other types of coupling possess this useful property. Where optimum gain and selectivity are required it can be shown that these will be obtained with a coupling about 80 per cent of the critical value—(Ref. C45). Greater selectivity can be achieved with less coupling than this value while increased gain will result from tighter coupling.

When the highest possible selectivity without serious loss of gain is desired from a pair of tuned coupled circuits, a practical compromise is to reduce coupling to 0.5 k_c at which value the gain is 0.8 times the optimum. The selectivity then approaches that which would be obtained by separating the two circuits with a valve (assuming this be done without altering Q' and Q_2). For other relationships between gain and selectivity, refer to Reed (Ref. C1) or to Aiken (Ref. C2).

(ii) Miscellaneous methods of coupling

There are other types of coupling which may be used between tuned circuits as alternatives to mutual inductance. Four such circuits are shown in Figs. 9.12, 9.13, 9.14 and 9.15. **High impedance or " top " coupling** is used in the circuits shown in Figures 9.12 and 9.13 and low impedance, or " bottom " coupling, is used in the circuits of Figures 9.14 and 9.15
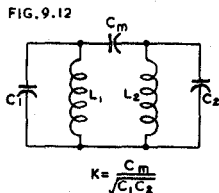
$$K = \frac{C_m}{\sqrt{C_1 C_2}}$$

Fig. 9.12. High impedance capacitive coupling.

$$K = \frac{\sqrt{L_1 L_2}}{L_m}$$
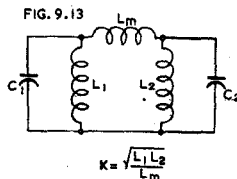
Fig. 9.13. High impedance inductive coupling.
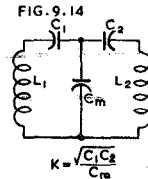
$$K = \frac{\sqrt{C_1 C_2}}{C_m}$$

Fig. 9.14. Low impedance capacitive coupling.

A fifth type is **link coupling** shown in Fig. 9.16A, in which a relatively small coupling inductance L_1' is coupled to L_1 and similarly L_2' to L_2 and L_1' is connected directly in series with L_2'. The behaviour of this circuit is the same as that to be described for Fig. 9.15.

" The coupling between two circuits, from a general point of view, is the relation between the possible rate of transfer of energy and the stored energy of the circuits " (Ref. C5, Sept. 1932).

From this definition it follows that for **low impedance coupling**

$$k = \frac{X_m}{\sqrt{(X_1 + X_m)(X_2 + X_m)}} \approx \frac{X_m}{\sqrt{X_1 X_2}} \text{when } X_1 \text{ and } X_2 \gg X_m$$

and that for **high impedance coupling**

$$k = \sqrt{\frac{X_1 X_2}{(X_1 + X_m)(X_2 + X_m)}} \approx \frac{\sqrt{X_1 X_2}}{X_m} \text{ when } X_1 \text{ and } X_2 \ll X_m$$

where X_m is the coupling reactance and X_1 and X_2 are the effective reactances of either the coils or the condensers* with which the circuits are tuned.

*X_1 and X_2 must be of the same " kind " (i.e. either inductive or capacitive) as X_m.

The effective reactances X_1 and X_2 in the high impedance case are calculated by regarding the actual tuning reactances ($L_1\omega_o$ and $L_2\omega_o$, or $1/C_1\omega_o$ and $1/C_2\omega_o$) as being in parallel with the coupling reactance X_m; while in the low impedance case, X_1 and X_2 are calculated by taking X_m to be in series with the actual tuning reactances.
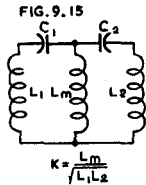

FIG. 9.15

$K = \dfrac{L_m}{\sqrt{L_1 L_2}}$

Fig. 9.15. *Low impedance induc- tive coupling.*

FIG. 9.16A

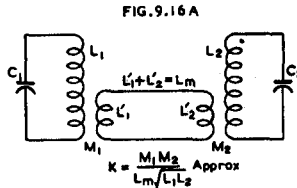$K = \dfrac{M_1 M_2}{L_m \sqrt{L_1 L_2}}$ Approx

Fig. 9.16A. *Link coupling.*

Mutual inductive coupling belongs to the low impedance coupling group; here $X_m = M\omega_o$, $X_1 = L_1\omega_o$, $X_2 = L_2\omega_o$; so that $k = M/\sqrt{L_1 L_2}$, in agreement with our previous definition in this particular case.

For a general analysis of the calculation of coupling coefficients see Chapter 26 Sect. 4(vii).

Application of the formulae given above to the circuits shown in Figures 9.12, 9.13, 9.14, 9.15 and 9.16A give the following results for the coefficient of coupling k:

Circuit	k (exact)	k (approximate)
Fig. 9.12	$\dfrac{C_m}{\sqrt{(C_1 + C_m)(C_2 + C_m)}}$	$\dfrac{C_m}{\sqrt{C_1 C_2}}$, when $C_m \ll (C_1, C_2)$
Fig. 9.13	$\sqrt{\dfrac{L_1 L_2}{(L_1 + L_m)(L_2 + L_m)}}$	$\dfrac{\sqrt{L_1 L_2}}{L_m}$ when $L_m \gg (L_1, L_2)$
Fig. 9.14	$\sqrt{\dfrac{C_1 C_2}{(C_1 + C_m)(C_2 + C_m)}}$	$\dfrac{\sqrt{C_1 C_2}}{C_m}$ when $C_m \gg (C_1, C_2)$
Fig. 9.15	$\dfrac{L_m}{\sqrt{(L_1 + L_m)(L_2 + L_m)}}$	$\dfrac{L_m}{\sqrt{L_1 L_2}}$, when $L_m \ll (L_1, L_2)$
Fig. 9.16A	$\dfrac{M_1 M_2}{L_m \sqrt{\left(L_1 - \dfrac{M_1{}^2}{L_m}\right)\left(L_2 - \dfrac{M_2{}^2}{L_m}\right)}}$ or $\dfrac{k_1 k_2}{\sqrt{(1 - k_1{}^2)(1 - k_2{}^2)}}$	$\dfrac{M_1 M_2}{L_m \sqrt{L_1 L_2}}$, when individual coup- lings are small. $k_1 k_2$, when individual coup- lings are small.

where $L_m = L_1' + L_2'$,

$$k_1 = \frac{M_1}{\sqrt{L_1 L_m}}, \text{ and } k_2 = \frac{M_2}{\sqrt{L_2 L_m}}.$$

When the coupling increases above k_t for the tuned circuits shown in Figs. 9.12, 9.13, 9.14, 9.15 and 9.16A, the two peaks in the response curve of the secondary move at unequal rates from the original resonant frequency (determined by $L_1 C_1 \omega_o{}^2 = L_2 C_2 \omega_o{}^2 = 1$). For the first four of these examples in the special case $L_1 = L_2$, $C_1 = C_2$, and for the fifth generally, one peak remains approximately stationary

(at ω_o), while the other peak moves to one side : the shift of the semi-stationary peak depends upon the series resistances of the two circuits and decreases with them, being zero in the ideal case $r_1 = r_2 = 0$: the second peak is lower in frequency in Figs. 9.12 and 9.15, but higher in Figs. 9.13, 9.14 and 9.16A. The selectivity and bandwidth (between peaks) may be calculated from the formulae (57) and (55) respectively—already quoted for transformer coupling—provided the appropriate value of k is used. It is theoretically possible, although seldom convenient, to combine two types of coupling in equal amounts to give symmetrical separation of the two peaks. Normally, when this is required simple mutual inductive coupling is used.

It may be shown that for all types of coupling the centre frequency is determined by the effective reactances obtained by taking the coupling reactance into account. Thus, for example, for Fig. 9.15, $(L_2 + L_m)C_2\omega_c^2 = 1$, where ω_c corresponds to the frequency of the minimum between the peaks ; while for Fig. 9.16A, $(L_2 - M_2^2/L_m)C_2\omega_c^2 = 1$.

(iii) Complex coupling

With any single type of coupling the gain and the band width vary with the frequency. Clearly, then, a single type of coupling cannot give satisfactory performance in the tuned radio frequency stages of a receiver where the frequency range is two or three to one. From the formulae already given it can be seen that for transformer coupling both the gain and the band width are approximately proportional to the frequency (assuming that Q_1 and Q_2 do not vary greatly) ; for other types of simple coupling k depends upon the square of the frequency, and hence band width and gain are functions of frequency.
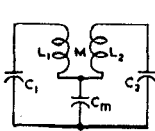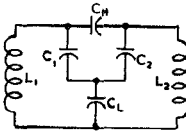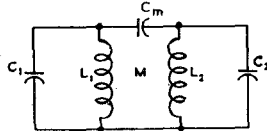

FIG. 9.16B   FIG. 9.16C   FIG. 9.16D

Fig. 9.16B. *Constant bandwidth using inductive and capacitive coupling.*
Fig. 9.16C. *Constant bandwidth using capacitive coupling.*
Fig. 9.16D. *Complex coupling with mutual inductive and top capacitive coupling.*

In practice, a reasonably constant band width over the tuning range can be obtained by a suitable combination of the types of coupling already described. Two common arrangements are shown in Figs. 9.16B and 9.16C.

For the circuit shown in Fig. 9.16B the coupling reactance $X_m = \omega M + 1/\omega C_m$, while for that shown in the Fig. 9.16C it is

$$\frac{1}{\omega C_L} + \frac{C_H}{\omega C_1 C_2}.$$

The corresponding coupling factors are respectively

$$k \approx \frac{M + 1/\omega^2 C_m}{\sqrt{L_1 L_2}} \qquad (k < 0.05),$$

and $k \approx \dfrac{C_H}{\sqrt{C_1 C_2}} + \dfrac{\sqrt{C_1 C_2}}{C_L} \qquad (k < 0.05)$.

Aiken (Ref. C2) gives a practical design procedure for obtaining the values of the components which give the best average results over the whole tuning range.

When k has been determined, the band width and selectivity may be calculated from eqns. (55) and (57) given earlier. Also, for circuits with the same values of Q, a suitable value of k is given by $kQ = 0.5$—as in the case of transformer coupling—when the greatest possible selectivity without notable loss of gain is desired.

In Fig. 9.16D there is a combination of mutual inductive coupling and top capacitive coupling, as commonly used in aerial and r-f coils, and, effectively, in i-f transformers. The analysis is given in Chapter 26 Sect. 4 (vii).

See also Sect. 11 Summary of Formulae.

SECTION 9 : RESPONSE OF IDENTICAL AMPLIFIER STAGES IN CASCADE

When two or more amplifier stages having identical circuits and values are connected in cascade, the overall gain is the product of the gains, and the resultant selectivity is the product of the selectivities of all the stages. For n stages, therefore, the **total gain is $A_o{}^n$** ; the **selectivity for single tuned circuits is**

$$(A/A_o)^n = (1 + Q^2Y^2)^{-n/2},$$

while **for coupled pairs**

$$\left(\frac{A}{A_o}\right)^n = \left[\left(1 - \frac{Q^2Y^2}{1 + k^2Q^2}\right)^2 + \left(\frac{2QY}{1 + k^2Q^2}\right)^2\right]^{-n/2}$$

When k^2Q^2 is very small, the selectivity of n coupled pairs is almost the same as that of $2n$ single tuned circuits. Thus there is a limit to the improvement of selectivity obtained by reduction of the coupling of coupled pairs of tuned circuits. When it is possible to increase Q, there is a corresponding improvement in selectivity. The tendency with several stages of single tuned circuits is to produce a very sharp peak at the centre frequency, which may seriously attenuate the higher audio frequencies of a modulated signal. Conditions are much better with coupled pairs, because two peaks with small separation appear as Q is increased, if the coupling is not too close. Difficulties occur when Q is increased so much that a deep " valley " or trough occurs between the peaks. The practical limit is usually a ratio of 1 : 1.5 overall gain between the response at the bottom of the valley and that at the two peaks. It is then good practice to add another stage employing a single tuned circuit which substantially removes the " valley " of the preceding circuits. The procedure by which the best results may be obtained is described by Ho-Shou Loh (Ref. C13). In this manner a nearly flat response may be obtained over a range of frequencies 10 Kc/s to 20 Kc/s wide, with very sharp discrimination against frequencies 20 Kc/s or more away from the centre frequency, 450 to 460 Kc/s.

See also Sect. 11 Summary of Formulae.

SECTION 10 : UNIVERSAL SELECTIVITY CURVES

In Figs. 9.17 and 9.18 are shown universal selectivity curves taken from Maynard's data (Ref. D2). These curves apply to a pair of coupled tuned circuits, and are not restricted to circuits of equal Q. Fig. 9.17 gives the gain at various frequencies off the centre frequency in terms of the gain at the centre frequency, for various coefficients of coupling ; the ordinate scale D is proportional to $Q\varDelta f_o/f_o$. The Q shown in Figs. 9.17 and 9.18 is Q_2 for all expressions containing a and b.

The phase difference between the currents in the primary and secondary circuits can readily be obtained from Fig. 9.18. There we have plotted an angle θ as a function of D for various coefficients of coupling, and the phase shift is $\theta \pm 90°$, the positive sign being taken when the coupling is negative, and the negative sign when k is positive.

The parameter b, which is a measure of the coupling, becomes simply k/k_c when the tuned circuits have equal values of Q ; also, in this case, the variable D becomes simply $2Q\varDelta f_o/f_o$. For convenience, scales have been added to give A_o/A in terms of $Q\varDelta f_o$ for a number of values of f_o for two identical coupled circuits.

In deriving the curves of Figs. 9.17 and 9.18 it was assumed that Q and k do not vary appreciably over the range considered, thus giving symmetrical selectivity curves ;

and that Q is reasonably high (> 25 say). Very low values of Q require a different curve for each value, but the effect is only to alter slightly the skirts of the curves without altering appreciably the main portions.

As an illustration of the use of Fig. 9.17 consider the example $f_0 = 1000$ Kc/s, $Q_1 = Q_2 = 200$, $k/k_c = 2 = b$: we see that the peak occurs at $Q \Delta f_0 \approx 800$, i.e. $\Delta f_0 = 4$ Kc/s, and that the gain in the valley compared with the gain at the peak $= 0.8$; these results agree well with those calculated from the formulae already given, namely
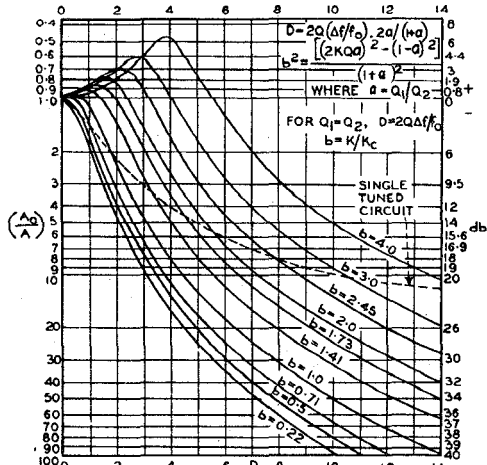

FIG. 9·17   UNIVERSAL SELECTIVITY CURVES FOR TWO COUPLED CIRCUITS

Band width is : $f_0 \sqrt{k^2 - k_c^2} \approx 8.6$ Kc/s.

Ratio of gain in valley to that at peak is :
$$\frac{2k/k_c}{1 + (k/k_c)^2} = \frac{2b}{1 + b^2} = \frac{4}{5}.$$

Phase change at $\Delta f_o = (+)$ 4 Kc/s : (from Fig. 9.18) phase change $\approx 20 \pm 90°$.

In conclusion, we give the selectivity curves of Fig. 9.19 to illustrate some of our remarks in preceding sections. These curves have been derived from those of Fig. 9.17, but here the abscissa is A_{opt}/A, where A_{opt} is the gain at the centre frequency where $k = k_c$.

These curves are for the case of two identical coupled circuits ; they show how the maximum gain, band width and depth of the valley between peaks vary with the ratio k/k_c.

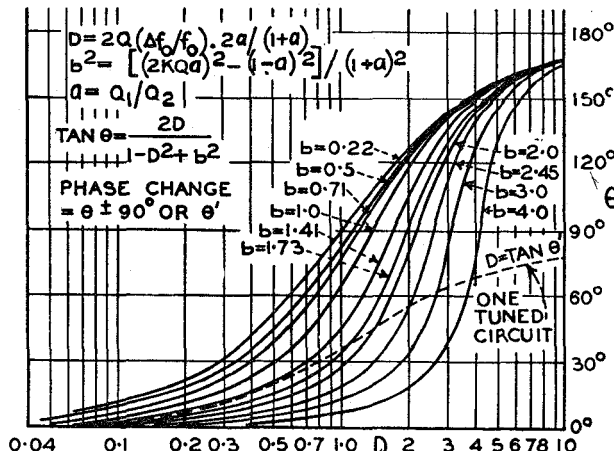In all expressions using a, the Q mentioned is the secondary Q, namely Q_2.

FIG. 9·18   UNIVERSAL PHASE SHIFT CURVES FOR TWO COUPLED CIRCUITS

SECTION 11 : SUMMARY OF FORMULAE FOR TUNED CIRCUITS

(1) NOMENCLATURE

L = inductance (in henrys unless otherwise stated)

C = capacitance (in farads unless otherwise stated)

f_o = resonant frequency (in cycles per second unless otherwise stated)

f_n = natural resonant frequency (in cycles per second unless otherwise stated)

f = frequency (in cycles/sec. unless otherwise stated)

$\Delta f_o = f - f_o$ (in cycles/sec.)

$Y = (f/f_o - f_o/f)$

π = 3.1416 approximately

k = coefficient of coupling

$\omega = 2\pi f$, $\omega_o = 2\pi f_o$, $\omega_n = 2\pi f_n$ radians per second

r = series resistance (in ohms)

r' = resistance of a series resonant circuit at resonance (in ohms)

R = shunt resistance (in ohms)

R_e = effective shunt resistance of a parallel resonant circuit at resonance when $R = \infty$ (in ohms)

R_D = resonant impedance (in ohms)

e = voltage across the circuit at a time t

E = initial voltage of charged condenser

ϵ = 2.718 (ϵ is the base of Naperian Logarithms)

t = time (in seconds)

α = damping factor

δ = logarithmic decrement

λ = wavelength in metres

i = current at frequency f (in amperes)

i_o = current at resonant frequency f_o

A = gain at frequency f

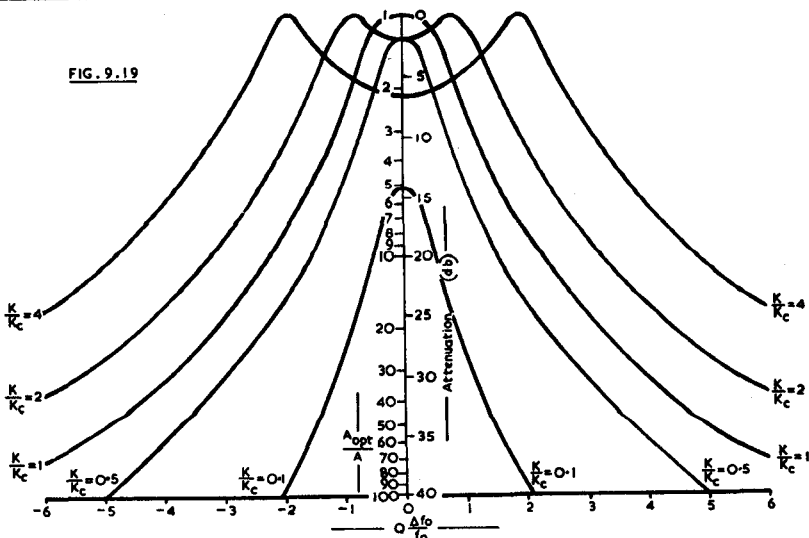A_o = gain at frequency f_o

Q = magnification factor.

FIG. 9.19

Selectivity Curves for Two Identical Coupled Circuits,
Showing the Variation in Maximum Gain,
Band Width and Depth of Valley with K/K_c

(2) NATURAL RESONANT FREQUENCY (f_n)

Exact formula

$$f_n = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{r^2}{4L^2}} \text{ cycles per second} \tag{1}$$

Approximate formula for use when r is small compared with $2\sqrt{L/C}$:—

$$f_n \approx f_o = \frac{1}{2\pi\sqrt{LC}}\text{c/s} \tag{2}$$

For numerical use this may be put in the form

$$f_n \approx f_o = \frac{159\,200}{\sqrt{LC}}\text{c/s, where } L \text{ is in microhenrys} \tag{3}$$
$$\text{and} \quad C \text{ is in microfarads ;}$$

or $\quad f_n \approx f_o = \dfrac{159\,200}{\sqrt{LC}}\text{Kc/s, where } L \text{ is in microhenrys} \hfill (4)$

$$\text{and} \quad C \text{ is in micromicrofarads.}$$

(3) WAVELENGTH (λ)

Wavelength (in metres) $= 1884\sqrt{LC},$ $\hfill (5)$

where L is in microhenrys and C is in microfarads.

Wavelength $\times$ frequency $= 2.9979 \times 10^8$ metres per second $\hfill (6)$
$$\approx 3 \times 10^8 \text{ metres per second.}$$

Note : Equations (5) and (6) are based on the velocity of electromagnetic radiation in a vacuum as determined by Dr. Essen and others in 1951 (see Ref. E1).

$$\text{Wavelength} \approx \frac{300\,000}{\text{frequency in Kc/s}} \approx \frac{300}{\text{frequency in Mc/s}} \tag{7}$$

(4) DAMPED OSCILLATIONS

$e = E\epsilon^{-\alpha t} \cos \omega_n t,$ where $\alpha = r/2L$ (damping factor) $\hfill (8)$

$\delta = r/2f_n L$ (logarithmic decrement) $\hfill (9)$

(5) SERIES RESONANCE

$L, C,$ and ω_o :

For resonance

$LC\omega_o^2 = 1$

$\omega_o L = 1/\omega_o C$ ohms

$\omega_o = 1/\sqrt{LC}$ radians/second

$\omega_o L = \sqrt{L/C}$ ohms

$\omega_o C = \sqrt{C/L}$ mhos.

L, C and f_o :

For resonance $\hfill (10)$

$2\pi f_o L = 1/2\pi f_o C$ ohms

$$LC = \frac{1}{39.48 f_o^2} \text{ (henrys} \times \text{farads)}$$

or $LC = \dfrac{2.533 \times 10^{10}}{f_o^2} \mu H \times \mu F$

where L is in microhenrys

and C is in microfarads.

$$Q = \omega_o L/r' = 1/r'C\omega_o, \tag{11}$$

where r' is the effective series resistance at resonance,

i.e. $\quad r' = r + L/CR$

where r is the series resistance

and R the parallel resistance.

$$\text{Therefore } Q = \frac{\omega_o L_1}{r + L/CR} = \frac{1}{r/\omega_o L + 1/RC\omega_o} \tag{12}$$

$$= \frac{1}{\omega_o Cr + 1/RC\omega_o} = \frac{1}{r\sqrt{C/L} + \sqrt{L/C}/R} \tag{13}$$

When $L/CR \ll r$: $Q \approx \omega_o L/r \approx 1/\omega_o Cr.$ $\hfill (14)$

Magnification factor (Q) : Ratio of the voltage across either reactance to the voltage across the circuit.

(6) PARALLEL RESONANCE

$$\text{At resonance, } \omega_o = \frac{1}{\sqrt{LC}} \cdot \frac{1}{\sqrt{1 + r^2/\omega_o^2 L^2}} \approx \frac{1}{\sqrt{LC}}. \tag{15}$$

$$Q = R_D/\omega_o L = R_D \omega_o C, \tag{16}$$

where R_D, the resonant impedance, is the effective shunt resistance of the circuit at resonance,

and $1/R_D = 1/R + Cr/L$.

Therefore $Q = \dfrac{1}{r\sqrt{C/L} + \sqrt{L/C}/R}$ etc., as in (12), (13) above.

When R is infinite (i.e. no shunt resistance),

$$R_D = R_e = L/Cr = Q/\omega_o C = \omega_o L Q = Q^2 r. \tag{17}$$

Magnification factor

Ratio of total circulating current to input current $= Q$ \hfill (18)

(7) SELECTIVITY

(a) Series Resonant Circuit

$$i_o/i = A_o/A = \sqrt{1 + Q^2(f/f_o - f_o/f)^2} = \sqrt{1 + Q^2 Y^2}. \tag{19}$$

$$\tan \phi = QY = Q\Delta f_o/f_o \cdot \frac{2 + \Delta f_o/f_o}{1 + \Delta f_o/f_o}; \tag{20}$$

i lags behind i_o when $f > f_o$, and leads i_o when $f < f_o$.

When $\Delta f_o/f_o$ is small,

$$i_o/i \approx \sqrt{1 + 4Q^2(\Delta f_o/f_o)^2}, \tag{21}$$

and $\tan \phi \approx 2Q\Delta f_o/f_o$. \hfill (22)

When i_o/i is large, $i_o/i \approx QY$. \hfill (23)

(b) Parallel Resonant Circuit

$$A_o/A = R_D/Z = i/i_o = \sqrt{1 + Q^2 Y^2} \tag{24}$$

$$\tan \phi = -QY; \tag{25}$$

where i and i_o are the total currents;

i leads i_o when $f > f_o$ and lags behind i_o when $f < f_o$.

When $\Delta f_o/f_o \ll 1$, $i/i_o \approx \sqrt{1 + 4Q^2 \Delta f_o^2/f_o^2}$. \hfill (26)

and $\tan \phi \approx -2Q\Delta f_o/f_o$. \hfill (27)

When $i/i_o \gg 1$, $i/i_o \approx QY$. \hfill (28)

(8) R-F TRANSFORMER, UNTUNED PRIMARY, TUNED SECONDARY

When the primary impedance can be neglected,

$$\text{Gain } A_o = \frac{g_m}{\dfrac{M}{r_p L_2} + \dfrac{1}{\omega_o M Q_2}} = \frac{g_m}{\dfrac{1}{kR_D}\sqrt{\dfrac{L_2}{L_1}} + \dfrac{k}{r_p}\sqrt{\dfrac{L_1}{L_2}}} \tag{29}$$

The gain may be expressed in a number of alternative forms, for example,

$$A_o = \frac{\mu \omega_o M Q_2}{r_p + \dfrac{(\omega_o M)^2 Q_2}{\omega_o L_2}} = \frac{\mu \omega_o M Q_2}{r_p + \dfrac{\omega_o^2 M^2}{r}} \tag{30}$$

where r is the series resistance of the secondary.

In the special case where $\omega_o^2 M^2/r \ll r_p$, we have

$$A_o \approx g_m \omega_o M Q_2. \tag{31}$$

Selectivity

To determine selectivity, the effect of the resistance reflected into the secondary should be taken into account when its value $\omega_o^2 M^2/r_p > 5\%$ of r. The effective series resistance r' is $r + \omega_o^2 M^2/r_p$; so that the effective value of Q_2, Q_2' say, is

$$Q_2' = \omega_o L_2/r' = \frac{1}{\dfrac{1}{Q_2} + \dfrac{k^2}{r_p} \cdot \omega_o L_1}. \tag{32}$$

The selectivity is then obtained from

$$A_o/A = \sqrt{1 + Q_2'^2 Y^2}. \tag{33}$$

(9) R.F. TRANSFORMER, TUNED PRIMARY, TUNED SECONDARY

Gain

The gain at resonance (A_o) is

$$A_o = \frac{g_m\sqrt{R'R_2}}{k\sqrt{Q'Q_2} + \dfrac{1}{k\sqrt{Q'Q_2}}}, \tag{34}$$

where $1/R' = 1/r_p + 1/R_1$ and $1/Q' = L_1\omega_o/R' = \omega_o L_1/r_p + 1/Q_1$.

Maximum gain occurs when $k = k_c = 1/\sqrt{Q'Q_2}$, the **critical coupling coefficient,** and is given by

$$A_o \text{ (max.)} = g_m\sqrt{R'R_2/2}. \tag{35}$$

Also, when $k = k_c/2$, the value of A_o is approximately 0.8 of A_o (max.).

Identical circuits $(L_1 = L_2, \ C_1 = C_2, \ R' = R_2 = R, \ Q' = Q_2 = Q)$

    Critical Coupling : $\ k_c = 1/Q.$ (36)

    Maximum gain (at resonance) $= g_m R/2.$ (37)

                    = half gain of a single tuned circuit.

    Band width between peaks $(f_1 \text{ and } f_2)$,

$$(f_2 - f_1)/f_o = \sqrt{k^2 - k_c^2}. \tag{38}$$

    Selectivity

$$A_o/A = \sqrt{\left[1 - \frac{Q^2Y^2}{1 + k^2Q^2}\right]^2 + \left[\frac{2QY}{1 + k^2Q^2}\right]^2} \tag{39}$$

    Gain at peaks, f_2 and f_1, is very closely equal to the optimum value A_o (max.).

Circuits of unequal Q

    Transitional coupling factor, k_t, is

$$k_t = \sqrt{\tfrac{1}{2}(1/Q'^2 + 1/Q_2^2)}. \tag{40}$$

    Band width between peaks is

$$(f_2 - f_1)/f_o = \sqrt{k^2 - k_t^2}. \tag{41}$$

    Selectivity

$$\frac{A_o}{A} \approx \sqrt{\left[1 - \frac{Q'Q_2Y^2}{1 + k^2Q'Q_2}\right]^2 + \left[\frac{2Y\sqrt{Q'Q_2}}{1 + k^2Q'Q_2}\right]^2} \tag{42}$$

    When $k^2Q^2 \gg 1$ and $\Delta f_o/f_o \ll 1$,

$$A_o/A \approx \sqrt{1 - 8\Delta f_o^2/f_o^2 k^2}. \tag{43}$$

(10) COUPLING COEFFICIENTS

High impedance coupling*

$$k \approx \frac{C_m}{\sqrt{C_1C_2}} \quad \text{for capacitive coupling (Fig. 9.12)} \tag{44}$$

$$= \frac{\sqrt{L_1L_2}}{L_m} \quad \text{for inductive coupling (Fig. 9.13)} \tag{45}$$

where C_m = coupling capacitance and L_m = coupling inductance.

Low impedance coupling*

$$k \approx \frac{\sqrt{C_1C_2}}{C_m} \quad \text{for capacitive coupling (Fig. 9.14)} \tag{46}$$

$$\approx \frac{L_m}{\sqrt{L_1L_2}} \quad \text{for inductive coupling (Fig. 9.15)} \tag{47}$$

where C_m = coupling capacitance and L_m = coupling inductance.

    Link coupling*

$$k \approx \frac{M_1M_2}{L_m\sqrt{L_1L_2}} \quad \text{(Fig. 9.16A)}, \tag{48}$$

where M_1 = mutual inductance between L_1 and L_m,

    and M_2 = mutual inductance between L_2 and L_m.

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* For exact values see table in Section 8.

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